

Characterizations of flag Hardy space via Riesz transforms, maximal functions and Littlewood–Paley theory

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Abstract

In this paper, we extend the classical characterization of Hardy spaces via maximal functions, the Littlewood–Paley square function, Lusin area integrals and the Riesz transforms to the setting of the flag Hardy space $H_F^1(\mathbb{R}^n \times \mathbb{R}^m)$. The novel ingredients in this extension include establishing an appropriate discrete Calderón reproducing formulae in the flag setting and a version of the Plancherel–Pólya inequalities for flag quadratic forms. Since it is unknown if the Hardy spaces $H_F^1(\mathbb{R}^n \times \mathbb{R}^m)$ have an atomic decomposition, we have to circumvent this useful tool by the introduction of certain Poisson maximal functions in the flag setting.

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1 Introduction

1.1 Background and statement of results

Classical Calderón–Zygmund singular integrals and the Hardy–Littlewood maximal operator commute with the usual dilations on \mathbb{R}^n , $\delta \cdot x = (\delta x_1, \dots, \delta x_n)$ for $\delta > 0$. This theory has been well studied and is by now well understood, see for example the monograph [27]. On the other hand, *product* Calderón–Zygmund singular integrals and the *strong* maximal function commute with the multi-parameter dilations on \mathbb{R}^n , $\delta \cdot x = (\delta_1 x_1, \dots, \delta_n x_n)$ for $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{R}_+^n$. Product Calderón–Zygmund theory has been studied, for example, in Gundy–Stein [13], R. Fefferman and Stein [4], R. Fefferman [5, 6, 7], Chang and R. Fefferman [1, 2, 3], Journé [18], Pipher [26]. More precisely, R. Fefferman and Stein [4] studied the L^p boundedness ($1 < p < \infty$) for the product convolution singular integral operators. Journé in [18] introduced non-convolution product singular integral operators and established the product $T1$ theorem and proved the $L^\infty \rightarrow \text{BMO}$ boundedness for such operators. The product Hardy space $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ was first introduced by Gundy and Stein [13]. Chang and R. Fefferman [1, 2, 3] developed the theory of atomic decomposition and established the dual space of the Hardy space $H^1(\mathbb{R}^n \times \mathbb{R}^m)$, namely the product $\text{BMO}(\mathbb{R}^n \times \mathbb{R}^m)$ space. However, in these works the underlying multi-parameter structure is explicit.

When the underlying multi-parameter structure is not explicit, but only *implicit* an appropriate L^p theory, with $1 < p < \infty$, has only recently been developed. To be precise, the flag multi-parameter structure on the Heisenberg group \mathbb{H}^n was studied in [20] and [21]. In these papers the authors obtained the L^p , $1 < p < \infty$, boundedness of Marcinkiewicz multipliers on the Heisenberg group. This is surprising since these multipliers are invariant under a two parameter group of dilations on $\mathbb{C}^n \times \mathbb{R}$, while there is *no* two parameter group of *automorphic* dilations on \mathbb{H}^n . Moreover, they show that Marcinkiewicz multiplier can be characterized by

the convolution operator of the form $f * K$ where, however, K is a flag convolution kernel. See Nagel, Ricci, and Stein [23] for flag singular integrals on the Euclidean space and applications on certain quadratic CR submanifolds of \mathbb{C}^n and Nagel, Ricci, Stein, and Wainger [24, 25] further generalized the theory of singular integrals with flag kernels to a more general setting, namely, homogeneous group. They proved that on a homogeneous group singular integral operators with flag kernels are bounded on L^p , $1 < p < \infty$, and form an algebra. See also [10, 11, 12] for related work.

At the endpoint estimates, it is natural to expect that Hardy space and BMO bounds are available. However, the lack of automorphic dilations underlies the failure of such multipliers to be in general bounded on the classical Hardy space H^1 and also precludes a pure product Hardy space theory on the Heisenberg group. This was the original motivation in [17] to develop a theory of *flag* Hardy spaces H_{flag}^p , $0 < p \leq 1$ on the Heisenberg group \mathbb{H}^n , that is, in a sense ‘intermediate’ between the classical Hardy spaces $H^p(\mathbb{H}^n)$ and the product Hardy spaces $H_{product}^p(\mathbb{C}^n \times \mathbb{R})$. The flag H^p theory on the Heisenberg group developed in [17] includes the discrete version of the Calderón reproducing formula associated with the given multi-parameter structure and the Plancherel–Pólya type inequality in this setting. They established the flag Hardy spaces $H_{flag}^p(\mathbb{H}^n)$ via the discrete Littlewood–Paley square function, and then studied the dual space $CMO_{flag}^p(\mathbb{H}^n)$ using the corresponding Carleson measures. The Calderón–Zygmund decomposition in terms of functions in $H_{flag}^p(\mathbb{H}^n)$ and interpolation had also been developed. They show that singular integrals with flag kernels, which include the aforementioned Marcinkiewicz multipliers, are bounded on $H_{flag}^p(\mathbb{H}^n)$, as well as from $H_{flag}^p(\mathbb{H}^n)$ to $L^p(\mathbb{H}^n)$, for $0 < p \leq 1$. Moreover, they constructed a singular integral with a flag kernel on the Heisenberg group, which is not bounded on the classical Hardy spaces $H^1(\mathbb{H}^n)$. Since, as pointed out in [17], the flag Hardy space $H_{flag}^p(\mathbb{H}^n)$ is contained in the classical Hardy space $H^p(\mathbb{H}^n)$, this counterexample implies that $H_{flag}^1(\mathbb{H}^n) \subsetneq H^1(\mathbb{H}^n)$.

It was well known that the classical Hardy spaces can be characterized by the Riesz transforms, maximal functions, the Littlewood–Paley square function and Lusin area integrals. Thus natural questions arise: Can one characterize the flag Hardy spaces by the Littlewood–Paley square function, the Lusin area integral, maximal functions and the Riesz transforms? The main purpose of this paper is to address these questions. To be precise, the main result of this paper, Theorem 1.6 below, extends the classical characterization via the Riesz transforms, maximal functions, the Littlewood–Paley square function and the Lusin area integrals to the more complicated case of the flag Hardy spaces $H_F^1(\mathbb{R}^n \times \mathbb{R}^m)$.

To state the main result requires several definitions. To begin with, we first introduce the Littlewood–Paley square function and Lusin area integrals associated with the flag structure on

$\mathbb{R}^n \times \mathbb{R}^m$. For this purpose, let $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^{n+m})$ with $\text{supp } \widehat{\psi^{(1)}} \subset \left\{ \xi : \frac{1}{2} \leq |\xi| \leq 2 \right\}$ and

$$\int_0^\infty |\widehat{\psi^{(1)}}(t\xi)|^2 \frac{dt}{t} = 1 \text{ for all } \xi \in \mathbb{R}^n \times \mathbb{R}^m \setminus \{(0,0)\}.$$

Let $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$ with $\text{supp } \widehat{\psi^{(2)}} \subset \left\{ \eta : \frac{1}{2} \leq |\eta| \leq 2 \right\}$ and

$$\int_0^\infty |\widehat{\psi^{(2)}}(s\eta)|^2 \frac{ds}{s} = 1 \text{ for all } \eta \in \mathbb{R}^m \setminus \{0\}.$$

We set

$$\psi_{t,s}(x, y) = \psi_t^{(1)} *_{\mathbb{R}^m} \psi_s^{(2)}(x, y) := \int_{\mathbb{R}^m} \psi_t^{(1)}(x, y - z) \psi_s^{(2)}(z) dz, \quad (1.1)$$

where $\psi_t^{(1)}(x, y) = t^{-(n+m)} \psi^{(1)}(\frac{x}{t}, \frac{y}{t})$ and $\psi_s^{(2)}(z) = s^{-m} \psi^{(2)}(\frac{z}{s})$.

Definition 1.1. For $f \in L^1(\mathbb{R}^{n+m})$, the Littlewood–Paley square function $g_F(f)$ is defined by

$$g_F(f)(x, y) = \left\{ \int_0^\infty \int_0^\infty \left| \psi_{t,s} * f(x, y) \right|^2 \frac{dt}{t} \frac{ds}{s} \right\}^{1/2},$$

where $\psi_{t,s}(x, y)$ is the same as in (1.1).

We now introduce the Lusin area integral associated with the flag structure.

Definition 1.2. For $f \in L^1(\mathbb{R}^{n+m})$, the Lusin area integral of f is defined by

$$S_F(f)(x, y) = \left\{ \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{m+1}} \chi_{t,s}(x - x_1, y - y_1) |\psi_{t,s} * f(x_1, y_1)|^2 \frac{dx_1 dt}{t^{n+m+1}} \frac{dy_1 ds}{s^{m+1}} \right\}^{1/2},$$

where $\chi_{t,s}(x, y) = \chi_t^{(1)} *_{\mathbb{R}^m} \chi_s^{(2)}(x, y)$, $\chi_t^{(1)}(x, y) = t^{-(n+m)} \chi^{(1)}(\frac{x}{t}, \frac{y}{t})$, $\chi_s^{(2)}(z) = s^{-m} \chi^{(2)}(\frac{z}{s})$, $\chi^{(1)}(x, y)$ and $\chi^{(2)}(z)$ are the indicator function of the unit balls of \mathbb{R}^{n+m} and \mathbb{R}^m , respectively.

To define maximal functions associated with the flag structure, we first introduce the following collection of functions that will be used to build the maximal functions.

Definition 1.3. Let $\phi(x, y) = \phi^{(1)} *_{\mathbb{R}^m} \phi^{(2)}(x, y)$, where $\phi^{(1)} \in \mathcal{S}(\mathbb{R}^{n+m})$ and $\phi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$ satisfying

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} \phi^{(1)}(x, y) dx dy = \int_{\mathbb{R}^m} \phi^{(2)}(z) dz = 1.$$

We denote $\mathcal{D}_F(\mathbb{R}^n \times \mathbb{R}^m)$ by the collection of all functions ϕ that satisfy the above conditions.

With the collection $\mathcal{D}_F(\mathbb{R}^n \times \mathbb{R}^m)$, the non-tangential maximal function has the following definition.

Definition 1.4. Let $\phi \in \mathcal{D}_F(\mathbb{R}^n \times \mathbb{R}^m)$. For each $f \in L^1(\mathbb{R}^{n+m})$, the non-tangential maximal function of f is defined by

$$M_\phi^*(f)(x, y) = \sup_{(x_1, y_1, t, s) \in \Gamma(x, y)} |\phi_{t,s} * f(x_1, y_1)|,$$

where $\phi_{t,s}(x, y) = \phi_t^{(1)} *_{\mathbb{R}^m} \phi_s^{(2)}(x, y)$, $\phi_t^{(1)}(x, y) = t^{-(n+m)} \phi^{(1)}(\frac{x}{t}, \frac{y}{t})$, $\phi_s^{(2)}(z) = s^{-m} \phi^{(2)}(\frac{z}{s})$ and $\Gamma(x, y) = \{(x_1, y_1, t, s) : |x - x_1| \leq t, |y - y_1| \leq t + s\}$.

Similarly, we define the radial maximal function as follows.

Definition 1.5. Let $\phi \in \mathcal{D}_F(\mathbb{R}^n \times \mathbb{R}^m)$. For any $f \in L^1(\mathbb{R}^{n+m})$, the radial maximal function of f is defined by

$$M_\phi^+(f)(x, y) = \sup_{t, s > 0} |\phi_{t,s} * f(x, y)|,$$

where $\phi_{t,s}(x, y)$ is defined as in Definition 1.4.

Next we use $R_j^{(1)}$ to denote the j -th Riesz transform on \mathbb{R}^{n+m} , $j = 1, 2, \dots, n+m$, and we use $R_k^{(2)}$ to denote the k -th Riesz transform on \mathbb{R}^m , $k = 1, 2, \dots, m$. Namely, we have that

$$R_j^{(1)} f(x) = \text{p.v.} \int_{\mathbb{R}^{n+m}} \frac{x_j - y_j}{|x - y|^{n+m+1}} f(y) dy, \quad x \in \mathbb{R}^{n+m}$$

and

$$R_k^{(2)} f(z) = \text{p.v.} \int_{\mathbb{R}^m} \frac{w_k - z_k}{|w - z|^{m+1}} f(w) dw, \quad z \in \mathbb{R}^m.$$

We set $R_{j,k} = R_j^{(1)} * R_k^{(2)}$, that is, $R_{j,k}$ is the composition of $R_j^{(1)}$ and $R_k^{(2)}$. Note that the flag structure is involved in $R_{j,k}$.

Here and throughout the paper, we use $\|h\|_1$ to denote the $L^1(\mathbb{R}^{n+m})$ norm of a function h . The goal and main result of this paper is as follows

Theorem 1.6. All the following norms

$$\|g_F(f)\|_1, \|S_F(f)\|_1, \|M_\phi^*(f)\|_1, \|M_\phi^+(f)\|_1, \sum_{j=1}^{n+m} \sum_{k=1}^m \|R_{j,k}(f)\|_1 + \|f\|_1$$

are equivalent for $f \in L^1(\mathbb{R}^{n+m})$.

With Theorem 1.6 it is natural to introduce the flag Hardy space $H_F^1(\mathbb{R}^n \times \mathbb{R}^m)$ as follows.

Definition 1.7. The flag Hardy spaces $H_F^1(\mathbb{R}^n \times \mathbb{R}^m)$ is defined to be the collection of $f \in L^1(\mathbb{R}^{n+m})$ such that $g_F(f) \in L^1(\mathbb{R}^{n+m})$. The norm of $H_F^1(\mathbb{R}^n \times \mathbb{R}^m)$ is defined by

$$\|f\|_{H_F^1(\mathbb{R}^n \times \mathbb{R}^m)} = \|g_F(f)\|_1.$$

As a consequence, we prove that, as in the classical case, the flag Hardy space $H_F^1(\mathbb{R}^n \times \mathbb{R}^m)$ can be characterized by the Riesz transforms, maximal functions, the Littlewood–Paley square function and the Lusin area integrals.

Remark 1.8. *We note that in the definitions of $g_F(f)$ and $M_\phi^+(f)$, the integration and supremum there are the same as those in the product setting, respectively. However, we point out that the flag structure is embedded in the definition of the functions $\psi_{t,s}$ and $\phi_{t,s}$ there respectively.*

Remark 1.9. *Based on the Riesz transform characterization of $H_F^1(\mathbb{R}^n \times \mathbb{R}^m)$, and the duality studied in [17], it is direct that the following two statements are equivalent.*

- (i) $\varphi \in \text{BMO}_F(\mathbb{R}^n \times \mathbb{R}^m)$;
- (ii) *There exist $g_{j,k} \in L^\infty(\mathbb{R}^{n+m})$, $j = 0, 1, \dots, n+m$, $k = 0, 1, \dots, m$, such that*

$$\varphi = \sum_{j=0}^{n+m} \sum_{k=0}^m R_{j,k}(g_{j,k}).$$

1.2 Structure and strategy of proof of the main result

In Section 2 we prove the equivalence between $\|g_F(f)\|_1$ and $\|S_F(f)\|_1$. We recall that in the classical case to show that the L^p norms, with $p \leq 1$, of the Littlewood–Paley square function and Lusin area integral are equivalent, the crucial tool is the sup-inf inequality, namely the Plancherel–Pólya type inequality. In order to establish such an inequality, one needs to develop the discrete Calderón reproducing formula. See [15] for more details in the setting of spaces of homogeneous type in the sense of Coifman and Weiss. In the present flag setting, to obtain the equivalence between the square function and Lusin area integral, we will first establish a discrete Calderón reproducing formula and then prove the Plancherel–Pólya type inequality associated with the flag structure. As a consequence, the equivalence between $\|g_F(f)\|_1$ and $\|S_F(f)\|_1$ will follow.

As the second step, we provide the equivalence between $\|S_F(f)\|_1$ and $\|M_\phi^*(f)\|_1$. It is well known that in the classical case the Lusin area integral of f belongs to L^p , $0 < p \leq 1$, if and only if f has an atomic decomposition and, similarly, the non-tangential maximal function of f belongs to L^p , $0 < p \leq 1$, if and only if f has an atomic decomposition. Thus, the atomic decomposition is a powerful tool to show the equivalent L^p norms, $0 < p \leq 1$ between the Lusin area integral of f and the non-tangential maximal function of f . However, in our case, it is not clear that if there is an atomic decomposition associated with the flag structure.

In this paper, we will apply a new approach to show the equivalence between $\|S_F(f)\|_1$ and $\|M_\phi^*(f)\|_1$. More precisely, we will introduce the Lusin area integral, the non-tangential maximal function, and the radial maximal function via flag Poisson integrals. To do this, we introduce

the flag Poisson kernel by

$$P(x, y) = P^{(1)} *_{\mathbb{R}^m} P^{(2)}(x, y) = \int_{\mathbb{R}^m} P^{(1)}(x, y - z) P^{(2)}(z) dz,$$

where

$$P^{(1)}(x, y) = \frac{c_{n+m}}{(1 + |x|^2 + |y|^2)^{(n+m+1)/2}} \quad \text{and} \quad P^{(2)}(z) = \frac{c_m}{(1 + |z|^2)^{(m+1)/2}}$$

are the classical Poisson kernels on \mathbb{R}^{n+m} and \mathbb{R}^m , respectively.

For any $f \in L^1(\mathbb{R}^{n+m})$, we define the flag Poisson integral of f by

$$u(x, y, t, s) := P_{t,s} * f(x, y),$$

where $P_{t,s}(x, y) = P_t^{(1)} *_{\mathbb{R}^m} P_s^{(2)}(x, y)$.

Since $P_{t,s}(x, y) \in L^1(\mathbb{R}^{n+m})$, it is easy to see that $u(x, y, t, s)$ is well-defined. Moreover, for any fixed t and s , $P_{t,s} * f$ is a bounded C^∞ function and the function $u(x, y, t, s)$ is harmonic in (x, y, t) and (y, s) , respectively.

We now define the flag Lusin area integral of u as follows.

Definition 1.10. For $f \in L^1(\mathbb{R}^{n+m})$ and $u(x, y, t, s) = P_{t,s} * f(x, y)$, $S_F(u)$, the flag Lusin area integral of $u(x, y, t, s)$ is defined by

$$S_F(u)(x, y) = \left\{ \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{m+1}} \chi_{t,s}(x - x_1, y - y_1) |t \nabla^{(1)} s \nabla^{(2)} u(x, y, t, s)|^2 \frac{dx_1 dt}{t^{n+m+1}} \frac{dy_1 ds}{s^{m+1}} \right\}^{\frac{1}{2}},$$

where $\chi_{t,s}(x, y)$ is the same as in Definition 1.2, $\nabla^{(1)} = (\partial_t, \partial_{x_1}, \dots, \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_m})$ and $\nabla^{(2)} = (\partial_s, \partial_{y_1}, \dots, \partial_{y_m})$.

Next, we define the non-tangential maximal function of u .

Definition 1.11. Let $f \in L^1(\mathbb{R}^{n+m})$, the non-tangential maximal function of u is defined by

$$u^*(x, y) = \sup_{(x_1, y_1, t, s) \in \Gamma(x, y)} |P_{t,s} * f(x_1, y_1)|,$$

where $\Gamma(x, y) = \{(x_1, y_1, t, s) : |x - x_1| \leq t, |y - y_1| \leq t + s\}$.

Similarly, the radial maximal function of u is given by the following definition.

Definition 1.12. Let $f \in L^1(\mathbb{R}^{n+m})$, the radial maximal function of u is defined by

$$u^+(x, y) = \sup_{t>0, s>0} |P_{t,s} * f(x, y)|.$$

In Section 3, we will show the following inequalities:

$$\|S_F(u)\|_1 \lesssim \|u^*\|_1 \lesssim \|u^+\|_1 \lesssim \sum_{j=1}^{n+m} \sum_{k=1}^m \|R_{j,k}(f)\|_1 + \|f\|_1.$$

In Section 4, the following estimates will be concluded:

- (I) $\|S_F(f)\|_1 \lesssim \|S_F(u)\|_1,$
- (II) $\|u^*\|_1 \approx \|M_\Phi^*(f)\|_1,$
- (III) $\|u^+\|_1 \approx \|M_\Phi^+(f)\|_1,$
- (IV) $\sum_{j=1}^{n+m} \sum_{k=1}^m \|R_{j,k}(f)\|_1 + \|f\|_1 \lesssim \|g_F(f)\|_1.$

Indeed, for each $f \in L^1(\mathbb{R}^{n+m})$, by $\|g_F(f)\|_1 \approx \|S_F(f)\|_1$ together with the above estimates, we have the following chain of inequalities: for $f \in L^1(\mathbb{R}^{n+m})$,

$$\begin{aligned} \|S_F(f)\|_1 &\lesssim \|S_F(u)\|_1 \lesssim \|u^*\|_1 \lesssim \|M_\Phi^*(f)\|_1 \lesssim \|u^*\|_1 \lesssim \|u^+\|_1 \lesssim \|M_\Phi^+(f)\|_1 \lesssim \|u^+\|_1 \\ &\lesssim \sum_{j=1}^{n+m} \sum_{k=1}^m \|R_{j,k}(f)\|_1 + \|f\|_1 \\ &\lesssim \|g_F(f)\|_1 \\ &\lesssim \|S_F(f)\|_1. \end{aligned}$$

This implies the main result, Theorem 1.6.

2 The equivalence of $\|g_F(f)\|_1$ and $\|S_F(f)\|_1$

2.1 Discrete Calderón reproducing formula

We first recall the following test function space $\widetilde{\mathcal{M}}_d$ with the size and smoothness conditions on \mathbb{R}^d for arbitrary positive integer d , which was introduced in [14].

Definition 2.1. Fix two exponents $0 < \beta < 1$ and $\gamma > 0$. We say that f defined on \mathbb{R}^d , belongs to $\widetilde{\mathcal{M}}_d(\beta, \gamma, r, x_0)$, $r > 0$ and $x_0 \in \mathbb{R}^d$, if

$$|f(x)| \leq C \frac{r^\gamma}{(r + |x - x_0|)^{d+\gamma}}, \quad (2.1)$$

$$|f(x) - f(x')| \leq C \left(\frac{|x - x'|}{r + |x - x_0|} \right)^\beta \frac{r^\gamma}{(r + |x - x_0|)^{d+\gamma}} \quad (2.2)$$

for $|x - x'| \leq \frac{r + |x - x_0|}{2}$. If $f \in \widetilde{\mathcal{M}}_d(\beta, \gamma, r, x_0)$, then the norm of f is defined by

$$\|f\|_{\widetilde{\mathcal{M}}_d(\beta, \gamma, r, x_0)} = \inf\{C : (2.1) \text{ and } (2.2) \text{ hold}\}.$$

Then we recall the test function space $\mathcal{M}_d \subset \widetilde{\mathcal{M}}_d$ on \mathbb{R}^d with a cancellation condition.

Definition 2.2. Fix two exponents $0 < \beta < 1$ and $\gamma > 0$. We say that f defined on \mathbb{R}^d , belongs to $\mathcal{M}_d(\beta, \gamma, r, x_0)$, $r > 0$ and $x_0 \in \mathbb{R}^d$, if $f \in \widetilde{\mathcal{M}}_d(\beta, \gamma, r, x_0)$ and

$$\int_{\mathbb{R}^d} f(x) dx = 0.$$

If $f \in \mathcal{M}_d(\beta, \gamma, r, x_0)$, then the norm of f is defined by

$$\|f\|_{\mathcal{M}_d(\beta, \gamma, r, x_0)} = \|f\|_{\widetilde{\mathcal{M}}_d(\beta, \gamma, r, x_0)}.$$

We now define the test function space on $\mathbb{R}^{n+m} \times \mathbb{R}^m$ as follows.

Definition 2.3. Fix two exponents $0 < \beta < 1$ and $\gamma > 0$. We say that f defined on $\mathbb{R}^{n+m} \times \mathbb{R}^m$ belongs to $\widetilde{\mathcal{M}}_{(n+m) \times m}(\beta, \gamma, r_1, r_2, x_0, y_0, z_0)$, $r_1, r_2 > 0$ and $(x_0, y_0, z_0) \in \mathbb{R}^{n+m} \times \mathbb{R}^m$, if for each fixed $z \in \mathbb{R}^m$, $f(\cdot, \cdot, z) \in \widetilde{\mathcal{M}}_{n+m}(\beta, \gamma, r_1, x_0, y_0)$ and for each $(x, y) \in \mathbb{R}^{n+m}$, $f(x, y, \cdot) \in \widetilde{\mathcal{M}}_m(\beta, \gamma, r_2, z_0)$ and satisfies the following conditions:

- (1) $\|f(\cdot, \cdot, z)\|_{\widetilde{\mathcal{M}}_{n+m}(\beta, \gamma, r_1, x_0, y_0)} \leq C \frac{r_2^\gamma}{(r_2 + |z - z_0|)^{m+\gamma}},$
- (2) $\|f(x, y, \cdot)\|_{\widetilde{\mathcal{M}}_m(\beta, \gamma, r_2, z_0)} \leq C \frac{r_1^\gamma}{(r_1 + |x - x_0| + |y - y_0|)^{n+m+\gamma}},$
- (3) $\|f(\cdot, \cdot, z) - f(\cdot, \cdot, z')\|_{\widetilde{\mathcal{M}}_{n+m}(\beta, \gamma, r_1, x_0, y_0)} \leq C \left(\frac{|z - z'|}{r_2 + |z - z_0|} \right)^\beta \frac{r_2^\gamma}{(r_2 + |z - z_0|)^{m+\gamma}}$
for $|z - z'| \leq \frac{r_2 + |z - z_0|}{2},$
- (4) $\|f(x, y, \cdot) - f(x', y', \cdot)\|_{\widetilde{\mathcal{M}}_m(\beta, \gamma, r_2, z_0)}$
 $\leq C \left(\frac{|x - x'| + |y - y'|}{r_1 + |x - x_0| + |y - y_0|} \right)^\beta \frac{r_1^\gamma}{(r_1 + |x - x_0| + |y - y_0|)^{n+m+\gamma}}$
for $|x - x'| + |y - y'| \leq \frac{r_1 + |x - x_0| + |y - y_0|}{2}.$

If $f \in \widetilde{\mathcal{M}}_{(n+m) \times m}(\beta, \gamma, r_1, r_2, x_0, y_0, z_0)$, the norm of f is defined by

$$\|f\|_{\widetilde{\mathcal{M}}_{(n+m) \times m}(\beta, \gamma, r_1, r_2, x_0, y_0, z_0)} = \inf\{C : (1) - (4) \text{ hold}\}.$$

Similarly we have the definition for the test function space $\mathcal{M}_{(n+m) \times m}(\beta, \gamma, r_1, r_2, x_0, y_0, z_0)$ as a subset in $\widetilde{\mathcal{M}}_{(n+m) \times m}(\beta, \gamma, r_1, r_2, x_0, y_0, z_0)$ and satisfies the corresponding cancellation conditions for the variables (x, y) and for z , respectively.

We would like to point out that if $f_1 \in \mathcal{M}_{n+m}(\beta, \gamma, r_1, x_0, y_0)$ and $f_2 \in \mathcal{M}_m(\beta, \gamma, r_2, z_0)$ then $f(x, y, z) = f_1(x, y)f_2(z) \in \mathcal{M}_{(n+m) \times m}(\beta, \gamma, r_1, r_2, x_0, y_0, z_0)$.

The flag test function is defined by:

Definition 2.4. Let $0 < \beta, \gamma < 1$, $r_1, r_2 > 0$ and $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^m$. We say that a function f defined on $\mathbb{R}^n \times \mathbb{R}^m$ belongs to the flag test function space $\widetilde{\mathcal{M}}_{flag}(\beta, \gamma, r_1, r_2, x_0, y_0)$ if there exists a function $f^\sharp(x, y, z) \in \widetilde{\mathcal{M}}_{(n+m) \times m}(\beta, \gamma, r_1, r_2, x_0, y_0, z_0)$ such that

$$f(x, y) = \int_{\mathbb{R}^m} f^\sharp(x, y - z, z) dz.$$

If $f \in \widetilde{\mathcal{M}}_{flag}(\beta, \gamma, r_1, r_2, x_0, y_0)$, the norm of f is defined by

$$\|f\|_{\widetilde{\mathcal{M}}_{flag}(\beta, \gamma, r_1, r_2, x_0, y_0)} = \inf \left\{ \|f^\sharp\|_{\mathcal{M}_{(n+m) \times m}(\beta, \gamma, r_1, r_2, x_0, y_0, z_0)} : f(x, y) = \int_{\mathbb{R}^m} f^\sharp(x, y - z, z) dz \right\}.$$

Similarly we can define the test function space $\mathcal{M}_{flag}(\beta, \gamma, r_1, r_2, x_0, y_0)$ with the flag cancellation condition as a subset in $\widetilde{\mathcal{M}}_{flag}(\beta, \gamma, r_1, r_2, x_0, y_0)$, which is projected from the product test function space $\mathcal{M}_{(n+m) \times m}(\beta, \gamma, r_1, r_2, x_0, y_0, z_0)$.

Observe that the flag structure is involved in the flag test function space $\mathcal{M}_{flag}(\beta, \gamma, r_1, r_2, x_0, y_0)$. We now prove the following discrete Calderón reproducing formula.

Theorem 2.5. Let $\psi_{t,s}$ be the same as in (1.1). Then there exist functions $\phi_{j,k}(x, y, x_I, y_J) \in \mathcal{M}_{flag}(\beta, \gamma, 2^{-j}, 2^{-k}, x_I, y_J)$ and a fixed large integer N such that for $f(x, y) = \int_{\mathbb{R}^m} f_1(x, y - z) f_2(z) dz$ with $f_1 \in \mathcal{M}_{n+m}(\beta, \gamma, r_1, r_2, x_0, y_0)$ and $f_2 \in \mathcal{M}_m(\beta, \gamma, r_3, z_0)$,

$$f(x, y) = \sum_j \sum_k \sum_I \sum_J |I||J| \phi_{j,k}(x, y, x_I, y_J) \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \psi_{t,s} * f(x_I, y_J) \frac{dt}{t} \frac{ds}{s}, \quad (2.3)$$

where the series converges in $L^2(\mathbb{R}^{n+m})$ and in the flag test function space and $I \subset \mathbb{R}^n$ and $J \subset \mathbb{R}^m$ are dyadic cubes with side-lengths $\ell(I) = 2^{-j-N}$ and $\ell(J) = 2^{-(j \wedge k - N)}$, x_I and y_J are any fixed points in I and J , respectively.

Note that for each $f \in L^1(\mathbb{R}^{n+m})$, $f \in (\mathcal{M}_{flag}(\beta, \gamma, r_1, r_2, x_0, y_0))'$. As a consequence of Theorem 2.5, by duality, if ψ is the same as in (1.1) and $f \in L^1(\mathbb{R}^{n+m})$,

$$\begin{aligned} & \langle f, \psi \rangle \\ &= \left\langle \sum_j \sum_k \sum_I \sum_J |I||J| \phi_{j,k}(\cdot, \cdot, x_I, y_J) \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \psi_{t,s} * f(x_I, y_J) \frac{dt}{t} \frac{ds}{s}, \psi \right\rangle. \end{aligned} \quad (2.4)$$

Remark 2.6. Indeed, the series in the right-hand side of (2.3) converges in the test function space $\mathcal{M}_{flag}(\beta, \gamma, r_1, r_2, x_0, y_0)$ and in the distribution space $(\mathcal{M}_{flag}(\beta, \gamma, r_1, r_2, x_0, y_0))'$, the dual of $\mathcal{M}_{flag}(\beta, \gamma, r_1, r_2, x_0, y_0)$. However, the proofs of such results are a little bit complicated. In this paper, we focus only on the Hardy space with $p = 1$. Thus, for our purpose, we only need the convergence in the distribution sense as given in (2.4).

Proof of Theorem 2.5. To show Theorem 2.5, observe that if $\psi_{t,s}$ are as in (1.1), by taking the Fourier transform, we have the following Calderón reproducing formula, namely for all $f \in L^2(\mathbb{R}^{n+m})$,

$$f(x, y) = \int_0^\infty \int_0^\infty \psi_{t,s} * \psi_{t,s} * f(x, y) \frac{dt}{t} \frac{ds}{s},$$

where the series converges in $L^2(\mathbb{R}^{n+m})$.

Suppose that $f \in \mathcal{M}_{flag}(\beta, \gamma, r_1, r_2, x_0, y_0)$ with $f(x, y) = \int_{\mathbb{R}^m} f_1(x, y - z) f_2(z) dz$ where $f_1 \in \mathcal{M}_{n+m}(\beta, \gamma, r_1, r_2, x_0, y_0)$ and $f_2 \in \mathcal{M}_m(\beta, \gamma, r_3, z_0)$. Applying Coifman's decomposition of the identity yields

$$\begin{aligned} f(x, y) &= \int_0^\infty \int_0^\infty \psi_{t,s} * \psi_{t,s} * f(x, y) \frac{dt}{t} \frac{ds}{s} \\ &= \sum_{j,k} \sum_{I,J} \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \int_{I \times J} \psi_{t,s}(x - u, y - v) \psi_{t,s} * f(u, v) du dv \frac{dt}{t} \frac{ds}{s} \\ &= \sum_{j,k} \sum_{I,J} \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \int_{I \times J} \psi_{t,s}(x - u, y - v) du dv \psi_{t,s} * f(x_I, y_J) \frac{dt}{t} \frac{ds}{s} \\ &\quad + \mathcal{R}(f)(x, y), \end{aligned} \tag{2.5}$$

where $I \subset \mathbb{R}^n$ and $J \subset \mathbb{R}^m$ are dyadic cubes with side-lengths $\ell(I) = 2^{-j-N}$ and $\ell(J) = 2^{-(j \wedge k - N)}$, x_I and y_J are **any fixed points** in I and J , respectively, and

$$\begin{aligned} \mathcal{R}(f)(x, y) &= \sum_{j,k} \sum_{I,J} \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \int_{I \times J} \psi_{t,s}(x - u, y - v) [\psi_{t,s} * f(u, v) - \psi_{t,s} * f(x_I, y_J)] du dv \frac{dt}{t} \frac{ds}{s}. \end{aligned}$$

Observing that

$$\begin{aligned} &\psi_{t,s} * f(u, v) - \psi_{t,s} * f(x_I, y_J) \\ &= \int_{\mathbb{R}^{n+m}} [\psi_{t,s}(u - u', v - v') - \psi_{t,s}(x_I - u', y_J - v')] f(u', v') du' dv', \end{aligned}$$

we can write

$$\begin{aligned} \mathcal{R}(f)(x, y) &= \int_{\mathbb{R}^{n+m}} \sum_{j,k} \sum_{I,J} \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \int_{I \times J} \psi_{t,s}(x - u, y - v) \\ &\quad \times [\psi_{t,s}(u - u', v - v') - \psi_{t,s}(x_I - u', y_J - v')] du dv \frac{dt}{t} \frac{ds}{s} f(u', v') du' dv'. \end{aligned}$$

Note that $\psi_{t,s}(x, y) = \int_{\mathbb{R}^m} \psi_t^{(1)}(x, y - z) \psi_s^{(2)}(z) dz$ and $f(x, y) = \int_{\mathbb{R}^m} f^\sharp(x, y - w, w) dw$ with $f^\sharp \in \mathcal{M}_{(n+m) \times m}(\beta, \gamma, r_1, r_2, x_0, y_0, z_0)$. Thus,

$$\mathcal{R}(f)(x, y) = \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \sum_{j,k} \sum_{I,J} \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \int_{I \times J} \psi_t^{(1)}(x - u, y - v - z) \psi_s^{(2)}(z)$$

$$\begin{aligned} & \times [\psi_t^{(1)}(u - u', v - w - v') - \psi_t^{(1)}(x_I - u', y_J - w - v')] \psi_s^{(2)}(w)] dw dv \\ & \frac{dt}{t} \frac{ds}{s} f^\sharp(u', v' - w', w') du' dv' dw' dz, \end{aligned}$$

We now define

$$\begin{aligned} \mathcal{R}^\sharp(f^\sharp)(x, y, z) &:= \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \sum_{j,k} \sum_{I,J} \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \int_{I \times J} \psi_t^{(1)}(x - u, y - v) \psi_s^{(2)}(z) \\ & \times [\psi_t^{(1)}(u - u', v - w - v') - \psi_t^{(1)}(x_I - u', y_J - w - v')] \psi_s^{(2)}(w)] dw dv du \\ & \frac{dt}{t} \frac{ds}{s} f^\sharp(u', v' - w', w') du' dv' dw' dz \end{aligned}$$

and then we can rewrite

$$\mathcal{R}(f)(x, y) = \int_{\mathbb{R}^m} \mathcal{R}^\sharp(f^\sharp)(x, y - z, z) dz.$$

We claim that if $f^\sharp(x, y, z) = f_1(x, y) f_2(z)$ with $f_1 \in \mathcal{M}_{n+m}(\beta, \gamma, r_1, r_2, x_0, y_0)$ and $f_2 \in \mathcal{M}_m(\beta, \gamma, r_3, z_0)$ then $\mathcal{R}^\sharp(f^\sharp)(x, y, z) \in \mathcal{M}_{(n+m) \times m}(\beta, \gamma, r_1, r_2, x_0, y_0, z_0)$ and

$$\|\mathcal{R}^\sharp(f^\sharp)\|_{\mathcal{M}_{(n+m) \times m}(\beta, \gamma, r_1, r_2, x_0, y_0, z_0)} \leq C 2^{-N} \|f^\sharp\|_{\mathcal{M}_{(n+m) \times m}(\beta, \gamma, r_1, r_2, x_0, y_0, z_0)}.$$

Assuming the claim for the moment, it implies that if $f(x, y) = \int_{\mathbb{R}^m} f_1(x, y - z) f_2(z) dz$ with $f_1 \in \mathcal{M}_{n+m}(\beta, \gamma, r_1, x_0, y_0)$ and $f_2 \in \mathcal{M}_m(\beta, \gamma, r_2, z_0)$ then $\mathcal{R}(f) \in \mathcal{M}_{flag}(\beta, \gamma, r_1, r_2, x_0, y_0)$ and

$$\|\mathcal{R}(f)\|_{\mathcal{M}_{flag}} \leq C 2^{-N} \|f\|_{\mathcal{M}_{flag}}.$$

Note that for $0 < \beta, \gamma < 1, t \sim 2^{-j}, s \sim 2^{-k}, x_I \in I$ and $y_J \in J$,

$$\frac{1}{|I||J|} \int_{I \times J} \psi_{t,s}(x - u, y - w) du dw = \int_{\mathbb{R}^m} \frac{1}{|I||J|} \int_{I \times J} \psi_t^{(1)}(x - u, y - w - z) du dw \psi_s^{(2)}(z) dz$$

is a flag test function in $\mathcal{M}_{flag}(\beta, \gamma, 2^{-j}, 2^{-k}, x_I, y_J)$ with

$$\frac{1}{|I||J|} \int_{I \times J} \psi_t^{(1)}(x - u, y - w - z) du dw \in \mathcal{M}_{n+m}(\beta, \gamma, 2^{-j}, x_I, y_J)$$

and

$$\psi_s^{(2)}(z) \in \mathcal{M}_m(\beta, \gamma, 2^{-k}, 0).$$

Therefore, if N is chosen large enough and by the above claim, we have

$$\left(\sum_{i=0}^{\infty} \mathcal{R}^i \left[\int_{I \times J} \psi_{t,s}(\cdot - u, \cdot - w) du dw \right] \right) (x, y) = |I||J| \phi_{j,k}(x, y, x_I, y_J),$$

where $\phi_{j,k}(x, y, x_I, y_J) \in \mathcal{M}_{flag}(\beta, \gamma, 2^{-j}, 2^{-k}, x_I, y_J)$ and hence,

$$f(x, y) = \sum_j \sum_k \sum_J \sum_I |I||J| \phi_{j,k}(x, y, x_I, y_J) \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} (\psi_{t,s} * f)(x_I, y_J) \frac{dt}{t} \frac{ds}{s}.$$

We now show the claim. To do this, we introduce the following.

Definition 2.7. Let T be a bounded linear operator on $L^2(\mathbb{R}^d)$ associated with a kernel $K(x, y)$ given by

$$Tf = \int_{\mathbb{R}^d} K(x, y)f(y)dy,$$

where $K(x, y)$ satisfies the following conditions: There exists a constant $C > 0$ such that

- (i) $|K(x, y)| \leq C|x - y|^{-d}$,
- (ii) $|K(x, y) - K(x', y)| \leq C|x - x'||x - y|^{-d-1}$ if $|x - x'| \leq |x - y|/2$,
- (iii) $|K(x, y) - K(x, y')| \leq C|y - y'||x - y|^{-d-1}$ if $|y - y'| \leq |x - y|/2$,
- (iv) $|K(x, y) - K(x', y) - K(x, y') + K(x', y')| \leq C|x - x'||y - y'||x - y|^{-d-2}$
if $|x - x'| \leq |x - y|/2$ and $|y - y'| \leq |x - y|/2$.

We denote by $\|K\|_{\mathbb{R}^d}$ the smallest constant C that satisfies (i)–(iv) above. The operator norm of T is defined by $\|T\| = \|T\|_{L^2 \rightarrow L^2} + \|K\|_{\mathbb{R}^d}$. Here we use d to denote arbitrary positive integer.

We would like to point out that the classical Calderón–Zygmund kernel $K(x, y)$ only needs to satisfy the conditions (i), (ii) and (iii). For our purpose, namely the boundedness of operators on test function space, condition (iv) is required, see [14] for the classical case. More precisely, we have the following

Lemma 2.8. Suppose that T is an operator as in Definition 2.7 and $T(1) = T^*(1) = 0$. Then T is bounded on the test function space $\mathcal{M}_d(\alpha, \beta, r, x_0)$ for $0 < \alpha, \beta < 1, r > 0$ and $x_0 \in \mathbb{R}^d$. Moreover, there exists a constant C such that

$$\|T(f)\|_{M_d(\alpha, \beta, r, x_0)} \leq C\|T\| \|f\|_{M_d(\alpha, \beta, r, x_0)}.$$

See [14] for the definition of $T(1) = T^*(1) = 0$ and the proof of Lemma 2.8. We now define the product operator as follows.

Definition 2.9. The operator T is said to be a product operator on $\mathbb{R}^{n+m} \times \mathbb{R}^m$ if T is bounded on $L^2(\mathbb{R}^{n+m} \times \mathbb{R}^m)$, and

$$Tf(x, y, z) = \int_{\mathbb{R}^{n+m+m}} K(x, y, z, u, v, w)f(u, v, w)dudvdw,$$

where $K(x, y, z, u, v, w)$, the kernel of T , satisfies the following conditions:

- (1) $\|K(\cdot, \cdot, z, \cdot, \cdot, w)\|_{\mathbb{R}^{n+m}} \leq C|z - w|^{-m}$,
- (2) $\|K(x, y, \cdot, u, v, \cdot)\|_{\mathbb{R}^m} \leq C(|x - u| + |y - v|)^{-(n+m)}$,
- (3) $\|K(\cdot, \cdot, z, \cdot, \cdot, w) - K(\cdot, \cdot, z', \cdot, \cdot, w)\|_{\mathbb{R}^{n+m}} \leq C \frac{|z - z'|}{|z - w|^{m+1}}$ for $|z - z'| \leq |z - w|/2$,

$$(4) \quad \|K(\cdot, \cdot, z, \cdot, \cdot, w) - K(\cdot, \cdot, z, \cdot, \cdot, w')\|_{\mathbb{R}^{n+m}} \leq C \frac{|w - w'|}{|z - w|^{m+1}} \quad \text{for } |w - w'| \leq |z - w|/2,$$

$$(5) \quad \|K(\cdot, \cdot, z, \cdot, \cdot, w) - K(\cdot, \cdot, z', \cdot, \cdot, w) - K(\cdot, \cdot, z, \cdot, \cdot, w) + K(\cdot, \cdot, z', \cdot, \cdot, w')\|_{\mathbb{R}^{n+m}} \leq C \frac{|z - z'| |w - w'|}{|z - w|^{m+2}} \\ \text{for } |z - z'| \leq |z - w|/2 \text{ and } |w - w'| \leq |z - w|/2,$$

$$(6) \quad \|K(x, y, \cdot, u, v, \cdot) - K(x', y', \cdot, u, v, \cdot)\|_{\mathbb{R}^m} \leq C \frac{|x - x'| + |y - y'|}{(|x - u| + |y - v|)^{n+m+1}} \\ \text{for } |x - x'| + |y - y'| \leq (|x - u| + |y - v|)/2,$$

$$(7) \quad \|K(x, y, \cdot, u, v, \cdot) - K(x, y, \cdot, u', v', \cdot)\|_{\mathbb{R}^m} \leq C \frac{|u - u'| + |v - v'|}{(|x - u| + |y - v|)^{n+m+1}} \\ \text{for } |u - u'| + |v - v'| \leq (|x - u| + |y - v|)/2,$$

$$(8) \quad \|K(x, y, \cdot, u, v, \cdot) - K(x', y', \cdot, u, v, \cdot) - K(x, y, \cdot, u', v', \cdot) + K(x', y', \cdot, u', v', \cdot)\|_{\mathbb{R}^m} \\ \leq C \frac{(|x - x'| + |y - y'|)(|u - u'| + |v - v'|)}{(|x - u| + |y - v|)^{n+m+2}} \\ \text{for } |x - x'| + |y - y'| \leq (|x - u| + |y - v|)/2 \text{ and } |u - u'| + |v - v'| \leq (|x - u| + |y - v|)/2.$$

We denote by $\|K\|$ the smallest constant C that satisfies (1)–(8) above. The operator norm of T is defined by $\|T\| = \|T\|_{L^2 \rightarrow L^2} + \|K\|$.

Proposition 2.10. *If T is a product operator as in Definition 2.9 and $T_1(1) = T_2(1) = T_1^*(1) = T_2^*(1) = 0$, then*

$$\|Tf\|_{\mathcal{M}_{n+m,m}(\beta, \gamma, r_1, r_2, x_0, y_0, z_0)} \leq C \|T\| \|f\|_{\mathcal{M}_{n+m,m}(\beta, \gamma, r_1, r_2, x_0, y_0, z_0)}$$

for all $f(x, y, z) = f_1(x, y)f_2(z)$ with $f_1 \in \mathcal{M}_{n+m}(\beta, \gamma, r_1, x_0, y_0)$ and $f_2 \in \mathcal{M}_m(\beta, \gamma, r_2, z_0)$.

See [18] for definitions of $T_1(1) = T_2(1) = T_1^*(1) = T_2^*(1) = 0$.

Remark 2.11. *Indeed, Proposition 2.10 holds for all $f \in \mathcal{M}_{n+m,m}(\beta, \gamma, r_1, r_2, x_0, y_0, z_0)$. The proof for such a result is a little bit complicated. However, Proposition 2.10 is enough to provide a proof for Theorem 2.5.*

We now prove Proposition 2.10. Suppose that $f(x, y, z) = f_1(x, y)f_2(z)$ with

$$\|f_1\|_{\mathcal{M}_{n+m}(\beta, \gamma, r_1, x_0, y_0)} = \|f_2\|_{\mathcal{M}_m(\beta, \gamma, r_2, z_0)} = 1.$$

We write

$$\begin{aligned} Tf(x, y, z) &= \int_{\mathbb{R}^{n+m+m}} K(x, y, z, u, v, w) f(u, v, w) du dv dw \\ &= \int_{\mathbb{R}^m} \int_{\mathbb{R}^{n+m}} K(x, y, z, u, v, w) f_1(u, v) du dv f_2(w) dw \\ &= \int_{\mathbb{R}^m} S(z, w) f_2(w) dw, \end{aligned}$$

where x, y and f_1 are fixed, and $S(z, w) = \int_{\mathbb{R}^{n+m}} K(x, y, z, u, v, w) f_1(u, v) du dv$.

We claim that for fixed $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $S(g)(z) = \int S(z, w) g(w) dw$ is an operator bounded on $\mathcal{M}_m(\beta, \gamma, r_2, z_0)$ with the kernel $S(z, w)$ satisfying Lemma 2.8. Moreover,

- (1) $|S(z, w)| \leq C |z - w|^{-m} |||T||| \frac{r_1^\gamma}{(r_1 + |x - x_0| + |y - y_0|)^{n+m+\gamma}},$
- (2) $|S(z, w) - S(z', w)| \leq C \frac{|z - z'|}{|z - w|^{m+1}} |||T||| \frac{r_1^\gamma}{(r_1 + |x - x_0| + |y - y_0|)^{n+m+\gamma}}$
for $|z - z'| \leq |z - w|/2$,
- (3) $|S(z, w) - S(z, w')| \leq C \frac{|w - w'|}{|z - w|^{m+1}} |||T||| \frac{r_1^\gamma}{(r_1 + |x - x_0| + |y - y_0|)^{n+m+\gamma}}$
for $|w - w'| \leq |z - w|/2$,
- (4) $|S(z, w) - S(z', w) - S(z, w') + S(z', w')|$
 $\leq C \frac{|z - z'| |w - w'|}{|z - w|^{m+2}} |||T||| \frac{r_1^\gamma}{(r_1 + |x - x_0| + |y - y_0|)^{n+m+\gamma}}$
for $|z - z'|, |w - w'| \leq |z - w|/2$,
- (5) $S(1) = S^*(1) = 0$.

The proof of the claim follows from Lemma 2.8. Indeed, for fixed $z, w \in \mathbb{R}^m$, the operator L with the kernel $K(x, y, z, u, v, w)$ is given by

$$L(f_1)(x, y, z, w) = \int_{\mathbb{R}^{n+m}} K(x, y, z, u, v, w) f_1(u, v) du dv.$$

By the condition (1) in Definition 2.9 together with Lemma 2.8, the operator L is bounded on $\mathcal{M}_{\mathbb{R}^{n+m}}(\beta, \gamma, r_1, x_0, y_0)$. Thus,

$$|L(f_1)(x, y, z, w)| \leq C |||T||| |z - w|^{-m} \frac{r_1^\gamma}{(r_1 + |x - x_0| + |y - y_0|)^{n+m+\gamma}},$$

which implies that $S(z, w)$ satisfies estimate (1) in the above claim, that is,

$$|S(z, w)| \leq C |z - w|^{-m} |||T||| \frac{r_1^\gamma}{(r_1 + |x - x_0| + |y - y_0|)^{n+m+\gamma}}.$$

Similarly, applying conditions (3) and (4) in Definition 2.9 together with Lemma 2.8, respectively, we conclude that $S(z, w)$ satisfies the estimates in (2) and (3) in the above claim, respectively. The condition (5) in Definition 2.9 together with Lemma 2.8 yields the estimate (5) in the above claim for $S(z, w)$.

Based on the estimates on $S(z, w)$, the kernel of S , applying Lemma 2.8 gives that the operator S is bounded on $\mathcal{M}_{\mathbb{R}^m}(\beta, \gamma, r_2, z_0)$ and hence

$$|Tf(x, y, z)| = |S(f_2)(z)| \leq C |||T||| \frac{r_1^\gamma}{(r_1 + |x - x_0| + |y - y_0|)^{n+m+\gamma}} \frac{r_2^\gamma}{(r_2 + |z - z_0|)^{m+\gamma}}$$

and

$$\begin{aligned} |Tf(x, y, z) - Tf(x, y, z')| &= |S(f_2)(z) - S(f_2)(z')| \\ &\leq C|||T||| \frac{r_1^\gamma}{(r_1 + |x - x_0| + |y - y_0|)^{n+m+\gamma}} \left(\frac{|z - z'|}{r_2 + |z - z_0|} \right)^\beta \frac{r_2^\gamma}{(r_2 + |z - z_0|)^{m+\gamma}} \end{aligned}$$

for $|z - z'| \leq \frac{r_2 + |z - z_0|}{2}$.

Similarly, if write

$$\begin{aligned} Tf(x, y, z) &= \int_{\mathbb{R}^{n+m+m}} K(x, y, z, u, v, w) f(u, v, w) dudvdw \\ &= \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^m} K(x, y, z, u, v, w) f_2(w) dw f_1(u, v) dudv \\ &= \int_{\mathbb{R}^{n+m}} R(x, y, z, u, v) f_1(u, v) dudv, \end{aligned}$$

where z and f_2 are fixed, and $R(x, y, z, u, v) = \int \int_{\mathbb{R}^m} K(x, y, z, u, v, w) f_2(w) dw$, then applying the same proof implies that the operator R is bounded on $\mathcal{M}_{\mathbb{R}^{n+m}}(\beta, \gamma, r_1, x_0, y_0)$ and moreover,

$$\begin{aligned} |Tf(x, y, z) - Tf(x', y', z)| &= |R(f_1)(x, y) - R(f_1)(x', y')| \\ &\leq C|||T||| \left(\frac{|x - x'| + |y - y'|}{r_1 + |x - x_0| + |y - y_0|} \right)^\beta \frac{r_1^\gamma}{(r_1 + |x - x_0| + |y - y_0|)^{n+m+\gamma}} \frac{r_2^\gamma}{(r_2 + |z - z_0|)^{m+\gamma}} \end{aligned}$$

for $|x - x'| + |y - y'| \leq (r_1 + |x - x_0| + |y - y_0|)/2$.

It remains to show the following estimate:

$$\begin{aligned} |Tf(x, y, z) - Tf(x', y', z) - Tf(x, y, z') + Tf(x', y', z')| & \\ &\leq C|||T||| \left(\frac{|x - x'| + |y - y'|}{r_1 + |x - x_0| + |y - y_0|} \right)^\beta \\ &\quad \times \left(\frac{|z - z'|}{r_2 + |z - z_0|} \right)^\beta \frac{r_1^\gamma}{(r_1 + |x - x_0| + |y - y_0|)^{n+m+\gamma}} \frac{r_2^\gamma}{(r_2 + |z - z_0|)^{m+\gamma}} \end{aligned}$$

for $|x - x'| + |y - y'| \leq (r_1 + |x - x_0| + |y - y_0|)/2$ and $|z - z'| \leq (r_2 + |z - z_0|)/2$.

To do this, write

$$\begin{aligned} Tf(x, y, z) - Tf(x, y, z') &= \int_{\mathbb{R}^{n+m+m}} [K(x, y, z, u, v, w) - K(x, y, z', u, v, w)] f(u, v, w) dudvdw \\ &= \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^m} [K(x, y, z, u, v, w) - K(x, y, z', u, v, w)] f_2(w) dw f_1(u, v) dudv \\ &= \int_{\mathbb{R}^{n+m}} H(x, y, z, z', u, v) f_1(u, v) dudv \\ &= H(f_1)(x, y, z, z'), \end{aligned}$$

where z, z' and f_2 are fixed, and

$$H(x, y, z, z', u, v) := \int_{\mathbb{R}^m} [K(x, y, z, u, v, w) - K(x, y, z', u, v, w)] f_2(w) dw.$$

We claim that the operator H with the kernel $H(x, y, z, z', u, v)$ defined above is bounded on $\mathcal{M}_{\mathbb{R}^{n+m}}(\beta, \gamma, r_1, x_0, y_0)$ and moreover,

$$\begin{aligned} & |Tf(x, y, z) - Tf(x', y', z) - Tf(x, y, z') + Tf(x', y', z')| \\ &= |H(f_1)(x, y, z, z') - H(f_1)(x', y', z, z')| \\ &\leq C|||T||| \left(\frac{|x - x'| + |y - y'|}{r_1 + |x - x_0| + |y - y_0|} \right)^\beta \left(\frac{|z - z'|}{r_2 + |z - z_0|} \right)^\beta \\ &\quad \times \frac{r_1^\gamma}{(r_1 + |x - x_0| + |y - y_0|)^{n+m+\gamma}} \frac{r_2^\gamma}{(r_2 + |z - z_0|)^{m+\gamma}} \end{aligned}$$

for $|x - x'| + |y - y'| \leq (r_1 + |x - x_0| + |y - y_0|)/2$ and $|z - z'| \leq (r_2 + |z - z_0|)/2$.

To see the claim, note first that by condition (2) in Definition 2.9 together with Lemma 2.8, for fixed x, y, u and v , the operator

$$\int_{\mathbb{R}^m} K(x, y, z, u, v, w) f_2(w) dw$$

is bounded on $\mathcal{M}_{\mathbb{R}^m}(\beta, \gamma, r_2, z_0)$ and hence, for $|z - z'| \leq (r_2 + |z - z_0|)/2$,

$$\begin{aligned} & \left| \int_{\mathbb{R}^m} [K(x, y, z, u, v, w) - K(x, y, z', u, v, w)] f_2(w) dw \right| \\ & \leq C(|x - u| + |y - v|)^{-(n+m)} \left(\frac{|z - z'|}{r_2 + |z - z_0|} \right)^\beta \frac{r_2^\gamma}{(r_2 + |z - z_0|)^{m+\gamma}}, \end{aligned}$$

which implies that for $|z - z'| \leq (r_2 + |z - z_0|)/2$,

$$|H(x, y, z, z', u, v)| \leq C(|x - u| + |y - v|)^{-(n+m)} \left(\frac{|z - z'|}{r_2 + |z - z_0|} \right)^\beta \frac{r_2^\gamma}{(r_2 + |z - z_0|)^{m+\gamma}}.$$

Write

$$\begin{aligned} & H(x, y, z, u, v) - H(x', y', z, u, v) \\ &= \int_{\mathbb{R}^m} [K(x, y, z, u, v, w) - K(x', y', z, u, v, w)] f_2(w) dw \\ &\quad - \int_{\mathbb{R}^m} [K(x, y, z', u, v, w) - K(x', y', z', u, v, w)] f_2(w) dw. \end{aligned}$$

By condition (6) in Definition 2.9 together with Lemma 2.8, for fixed x, y, x', y', u and v with $|x - x'| + |y - y'| \leq (|x - u| + |y - v|)/2$, the operator

$$\int_{\mathbb{R}^m} [K(x, y, z, u, v, w) - K(x', y', z, u, v, w)] f_2(w) dw$$

is bounded on $\mathcal{M}_{\mathbb{R}^m}(\beta, \gamma, r_2, z_0)$ and hence, for $|z - z'| \leq (r_2 + |z - z_0|)/2$ and $|x - x'| + |y - y'| \leq (|x - u| + |y - v|)/2$,

$$|H(x, y, z, u, v) - H(x', y', z', u, v)| \leq C \frac{|x - x'| + |y - y'|}{(|x - u| + |y - v|)^{n+m+1}} \left(\frac{|z - z'|}{r_2 + |z - z_0|} \right)^\beta \frac{r_2^\gamma}{(r_2 + |z - z_0|)^{m+\gamma}}.$$

Similarly, for $|z - z'| \leq (r_2 + |z - z_0|)/2$ and $|u - u'| + |v - v'| \leq (|x - u| + |y - v|)/2$,

$$|H(x, y, z, u, v) - H(x, y, z', u', v')| \leq C \frac{|u - u'| + |v - v'|}{(|x - u| + |y - v|)^{n+m+1}} \left(\frac{|z - z'|}{r_2 + |z - z_0|} \right)^\beta \frac{r_2^\gamma}{(r_2 + |z - z_0|)^{m+\gamma}}.$$

Finally, we write

$$\begin{aligned} & H(x, y, z, z', u, v) - H(x', y', z', u, v) - H(x, y, z, z', u', v') + H(x', y', z', u', v') \\ &= \int_{\mathbb{R}^m} [K(x, y, z, u, v, w) - K(x', y', z, u, v, w) - K(x, y, z, u', v', w) + K(x', y', z, u', v', w)] f_2(w) dw \\ &\quad - \int_{\mathbb{R}^m} [K(x, y, z', u, v, w) - K(x', y', z', u, v, w) \\ &\quad \quad - K(x, y, z', u', v', w) + K(x', y', z', u', v', w)] f_2(w) dw. \end{aligned}$$

Applying condition (7) in Definition 2.9 together with Lemma 2.8, for fixed x, y, x', y', u, v, u' and v' with $|x - x'| + |y - y'| \leq (|x - u| + |y - v|)/2$ and $|u - u'| + |v - v'| \leq (|x - u| + |y - v|)/2$, the operator

$$\int_{\mathbb{R}^m} [K(x, y, z, u, v, w) - K(x', y', z, u, v, w) - K(x, y, z, u', v', w) + K(x', y', z, u', v', w)] f_2(w) dw$$

is bounded on $\mathcal{M}_{\mathbb{R}^m}(\beta, \gamma, r_2, z_0)$ and hence, for $|z - z'| \leq (r_2 + |z - z_0|)/2$, $|x - x'| + |y - y'| \leq (|x - u| + |y - v|)/2$ and $|u - u'| + |v - v'| \leq (|x - u| + |y - v|)/2$,

$$\begin{aligned} & |H(x, y, z, u, v) - H(x', y', z', u, v)| \\ & \leq C \frac{(|x - x'| + |y - y'|)(|u - u'| + |v - v'|)}{(|x - u| + |y - v|)^{n+m+2}} \left(\frac{|z - z'|}{r_2 + |z - z_0|} \right)^\beta \frac{r_2^\gamma}{(r_2 + |z - z_0|)^{m+\gamma}}. \end{aligned}$$

Therefore, the operator

$$\int_{\mathbb{R}^m} [K(x, y, z, u, v, w) - K(x', y', z, u, v, w) - K(x, y, z, u', v', w) + K(x', y', z, u', v', w)] f_2(w) dw$$

is bounded on $\mathcal{M}_{\mathbb{R}^{n+m}}(\beta, \gamma, r_1, x_0, y_0)$ and this yields the claim. The proof of Proposition 2.10 is concluded. \square

2.2 Plancherel–Pólya type inequalities

Applying the discrete Calderón reproducing formula in (2.5) provides the following Plancherel–Pólya-type inequalities.

Theorem 2.12. *Suppose ψ is as in (1.1). Then for $f \in L^1(\mathbb{R}^{n+m})$,*

$$\left\| \left\{ \sum_j \sum_k \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \sum_J \sum_I \sup_{\substack{u \in I \\ v \in J}} |\psi_{t,s} * f(u, v)|^2 \chi_I(x) \chi_J(y) \frac{dt}{t} \frac{ds}{s} \right\}^{\frac{1}{2}} \right\|_1$$

$$\approx \left\| \left\{ \sum_j \sum_k \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \sum_J \sum_I \inf_{\substack{u \in I \\ v \in J}} |\psi_{t,s} * f(u,v)|^2 \chi_I(x) \chi_J(y) \frac{dt}{t} \frac{ds}{s} \right\}^{\frac{1}{2}} \right\|_1,$$

where $I \subset \mathbb{R}^n, J \subset \mathbb{R}^m$, are the same as in Theorem 2.5 and χ_I and χ_J are indicator functions of I and J , respectively.

Proof. For $f \in L^1(\mathbb{R}^{n+m})$, by Theorem 2.5,

$$\begin{aligned} & \psi_{t,s} * f(u,v) \\ &= \sum_{j'} \sum_{k'} \sum_{I'} \sum_{J'} |I'| |J'| \psi_{t,s} * \phi_{j',k'}(u,v) \int_{2^{-k'-N}}^{2^{-k'-N+1}} \int_{2^{-j'-N}}^{2^{-j'-N+1}} \psi_{t',s'} * f(x_{I'}, y_{J'}) \frac{dt'}{t'} \frac{ds'}{s'}. \end{aligned}$$

For $2^{-j-N} < t < 2^{-k-N+1}$ and $2^{-k-N} < s < 2^{-k-N+1}$, applying the classical almost orthogonal estimate yields that for $0 < \beta, \gamma < 1$,

$$|\psi_{t,s} * \phi_{j',k'}(u,v)| \leq C_N 2^{-|j-j'|\beta} 2^{-|k-k'|\beta} \frac{2^{-(j \wedge j')\gamma}}{(2^{-(j \wedge j')} + |x_{I'} - u|)^{n+\gamma}} \frac{2^{-[(k \wedge k') \wedge (j \wedge j')]\gamma}}{(2^{-[(k \wedge k') \wedge (j \wedge j')]} + |x_{J'} - v|)^{m+\gamma}}.$$

Observe that

$$\begin{aligned} & \sum_{I'} \sum_{J'} |I'| |J'| \frac{2^{-(j \wedge j')\gamma}}{(2^{-(j \wedge j')} + |x_{I'} - u|)^{n+\gamma}} \frac{2^{-[(k \wedge k') \wedge (j \wedge j')]\gamma}}{(2^{-[(k \wedge k') \wedge (j \wedge j')]} + |x_{J'} - v|)^{m+\gamma}} \\ & \quad \times \int_{2^{-k'-N}}^{2^{-k'-N+1}} \int_{2^{-j'-N}}^{2^{-j'-N+1}} \psi_{t',s'} * f(x_{I'}, y_{J'}) \frac{dt'}{t'} \frac{ds'}{s'} \\ & \leq C \left\{ M_s \left(\int_{2^{-k'-N}}^{2^{-k'-N+1}} \int_{2^{-j'-N}}^{2^{-j'-N+1}} \sum_{I'} \sum_{J'} \psi_{t',s'} * f(x_{I'}, y_{J'}) \chi_{I'} \chi_{J'} \frac{dt'}{t'} \frac{ds'}{s'} \right)^r (u,v) \right\}^{1/r}, \end{aligned}$$

where M_s is the strong maximal function on $\mathbb{R}^n \times \mathbb{R}^m$ and $\frac{n+m}{n+m+\beta} < r < 1$. See [9] for the proof of the classical case. Note that $x_{I'}$ and $y_{J'}$ are arbitrary points in I' and J' , respectively, we have

$$\begin{aligned} & \sup_{\substack{u \in I \\ v \in J}} |\psi_{t,s} * f(u,v)| \\ & \leq C_N \sum_{j'} \sum_{k'} 2^{-|j-j'|\beta} 2^{-|k-k'|\beta} 2^{-j'n(1-\frac{1}{r})} 2^{[(j \wedge j')-j']n(1-\frac{1}{r})} 2^{-k'n(1-\frac{1}{r})} 2^{[(k \wedge k')-k']m(1-\frac{1}{r})} \\ & \quad \times \left\{ M_s \left(\int_{2^{-k'-N}}^{2^{-k'-N+1}} \int_{2^{-j'-N}}^{2^{-j'-N+1}} \sum_{I'} \sum_{J'} \inf_{\substack{u' \in I' \\ v' \in J'}} \psi_{t',s'} * f(u',v') \chi_{I'} \chi_{J'} \frac{dt'}{t'} \frac{ds'}{s'} \right)^r (u,v) \right\}^{1/r}. \end{aligned}$$

Applying Hölder's inequality together with the facts that

$$\sum_j \sum_k 2^{-|j-j'|\beta} 2^{-|k-k'|\beta} 2^{-j'n(1-\frac{1}{r})} 2^{[(j \wedge j')-j']n(1-\frac{1}{r})} 2^{-k'n(1-\frac{1}{r})} 2^{[(k \wedge k')-k']m(1-\frac{1}{r})} \leq C,$$

$$\sum_J \sum_I \chi_I(x) \chi_J(y) \leq C$$

and

$$\int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \frac{dt}{t} \frac{ds}{s} \leq C$$

gives

$$\begin{aligned} & \left\| \left\{ \sum_j \sum_k \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \sum_J \sum_I \sup_{\substack{u \in I \\ v \in J}} |\psi_{t,s} * f(u,v)|^2 \chi_I(x) \chi_J(y) \frac{dt}{t} \frac{ds}{s} \right\}^{\frac{1}{2}} \right\|_1 \\ & \lesssim \left\| \sum_{j'} \sum_{k'} \left\{ M_s \left(\int_{2^{-k'-N}}^{2^{-k'-N+1}} \int_{2^{-j'-N}}^{2^{-j'-N+1}} \sum_{I'} \sum_{J'} \inf_{\substack{u' \in I' \\ v' \in J'}} |\psi_{t',s'} * f(u',v')| \chi_{I'} \chi_{J'} \frac{dt'}{t'} \frac{ds'}{s'} \right)^r (u,v) \right\}^{\frac{1}{r}} \right\|_1. \end{aligned}$$

By the Fefferman-Stein vector-valued maximal function inequality with $r < 1$, we get

$$\begin{aligned} & \left\| \left\{ \sum_j \sum_k \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \sum_J \sum_I \sup_{\substack{u \in I \\ v \in J}} |\psi_{t,s} * f(u,v)|^2 \chi_I(x) \chi_J(y) \frac{dt}{t} \frac{ds}{s} \right\}^{\frac{1}{2}} \right\|_1 \\ & \approx \left\| \left\{ \sum_j \sum_k \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \sum_J \sum_I \inf_{\substack{u \in I \\ v \in J}} |\psi_{t,s} * f(u,v)|^2 \chi_I(x) \chi_J(y) \frac{dt}{t} \frac{ds}{s} \right\}^{\frac{1}{2}} \right\|_1. \end{aligned}$$

The proof is completed. \square

We remark that applying the similar proof, for any fixed constant C_0 one can get the following Plancherel-Pólya-type inequalities:

$$\begin{aligned} & \left\| \left\{ \sum_j \sum_k \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \sum_J \sum_I \sup_{\substack{u \in C_0 I \\ v \in C_0 J}} |\psi_{t,s} * f(u,v)|^2 \chi_I(x) \chi_J(y) \frac{dt}{t} \frac{ds}{s} \right\}^{\frac{1}{2}} \right\|_1 \\ & \approx \left\| \left\{ \sum_j \sum_k \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \sum_J \sum_I \inf_{\substack{u \in I \\ v \in J}} |\psi_{t,s} * f(u,v)|^2 \chi_I(x) \chi_J(y) \frac{dt}{t} \frac{ds}{s} \right\}^{\frac{1}{2}} \right\|_1, \quad (2.6) \end{aligned}$$

where $C_0 I \subset \mathbb{R}^n$ and $C_0 J \subset \mathbb{R}^m$, are cubes with side-length $\ell(C_0 I) = C_0 2^{-j-N}$ and $\ell(C_0 J) = C_0 \ell(J) = 2^{-(j-N \wedge k-N)}$, respectively.

2.3 The equivalence of $\|g_F(f)\|_1$ and $\|S_F(f)\|_1$

2.3.1 The proof that $\|S_F(f)\|_1 \lesssim \|g_F(f)\|_1$

We write

$$\begin{aligned} \|S_F(f)(x, y)\|_1 &= \left\| \left\{ \sum_{j,k} \sum_{I,J} \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \chi_{t,s}(x - x_1, y - y_1) \right. \right. \\ & \quad \left. \left. \times |\psi_{t,s} * f(x_1, y_1)|^2 \chi_I(x) \chi_J(y) \frac{dx_1 dt}{t^{n+m+1}} \frac{dy_1 ds}{s^{m+1}} \right\}^{1/2} \right\|_1. \end{aligned}$$

where N is a fixed large integer as in the Plancherel-Pólya-type inequalities and $I \subset \mathbb{R}^n$ and $J \subset \mathbb{R}^m$ are dyadic cubes with side-length $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-(j-N \wedge k-N)}$, and χ_I and χ_J are indicator functions of I and J , respectively.

Observe that there exists a fixed constant C_0 such that for $2^{-j-N} \leq t \leq 2^{-j-N+1}$, $2^{-k-N} \leq s \leq 2^{-k-N+1}$ and $x_1 \in \mathbb{R}^n$ and $y_1 \in \mathbb{R}^m$,

$$\begin{aligned} & \chi_{t,s}(x - x_1, y - y_1) |\psi_{t,s} * f(x_1, y_1)|^2 \chi_I(x) \chi_J(y) \\ & \leq \chi_{t,s}(x - x_1, y - y_1) \sup_{\substack{u \in C_0 I \\ v \in C_0 J}} |\psi_{t,s} * f(u, v)|^2 \chi_I(x) \chi_J(y). \end{aligned}$$

Therefore,

$$\begin{aligned} \|S_F(f)(x, y)\|_1 & \leq \left\| \left\{ \sum_{j,k} \sum_{I,J} \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \chi_{t,s}(x - x_1, y - y_1) \right. \right. \\ & \quad \times \sup_{\substack{u \in C_0 I \\ v \in C_0 J}} |\psi_{t,s} * f(u, v)|^2 \chi_I(x) \chi_J(y) \frac{dx_1 dt}{t^{n+m+1}} \frac{dy_1 ds}{s^{m+1}} \left. \right\}^{1/2} \Big\|_1. \end{aligned}$$

Applying the estimate $\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \chi_{t,s}(x - x_1, y - y_1) dx_1 dy_1 \leq C t^{n+m} s^n$ together with the Plancherel-Pólya-type inequalities in (2.6) yields

$$\begin{aligned} & \|S_F(f)(x, y)\|_1 \\ & \leq \left\| \left\{ \sum_{j,k} \sum_{I,J} \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \sup_{\substack{u \in C_0 I \\ v \in C_0 J}} |\psi_{t,s} * f(u, v)|^2 \chi_I(x) \chi_J(y) \frac{dt ds}{t s} \right\}^{1/2} \right\|_1 \\ & \lesssim \left\| \left\{ \sum_{j,k} \sum_{I,J} \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} |\psi_{t,s} * f(x, y)|^2 \chi_I(x) \chi_J(y) \frac{dt ds}{t s} \right\}^{1/2} \right\|_1 \\ & = \|g_F(f)\|_1. \end{aligned}$$

2.3.2 The proof that $\|g_F(f)\|_1 \lesssim \|S_F(f)\|_1$

The proof of this part is similar. To see this, write

$$\begin{aligned} \|g_F(f)\|_1 & = \left\| \left\{ \int_0^\infty \int_0^\infty |\psi_{t,s} * f(x, y)|^2 \frac{dt ds}{t s} \right\}^{1/2} \right\|_1 \\ & = \left\| \left\{ \sum_{j,k} \sum_{I,J} \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} |\psi_{t,s} * f(x, y)|^2 \chi_I(x) \chi_J(y) \frac{dt ds}{t s} \right\}^{1/2} \right\|_1. \end{aligned}$$

By the Plancherel-Pólya-type inequalities in (2.6), the last term above is dominated by

$$C \left\| \left\{ \sum_{j,k} \sum_{I,J} \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \inf_{\substack{u \in C_0 I \\ v \in C_0 J}} |\psi_{t,s} * f(u, v)|^2 \chi_I(x) \chi_J(y) \frac{dt ds}{t s} \right\}^{1/2} \right\|_1$$

$$\begin{aligned}
&\leq C \left\| \left\{ \sum_{j,k} \sum_{I,J} \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \chi_{t,s}(x-x_1, y-y_1) \times \right. \right. \\
&\quad \left. \left. \inf_{\substack{u \in C_0 I \\ v \in C_0 J}} |\psi_{t,s} * f(u,v)|^2 \chi_I(x) \chi_J(y) \frac{dx_1 dt}{t^{n+m+1}} \frac{dy_1 ds}{s^{m+1}} \right\}^{1/2} \right\|_1 \\
&\leq C \|S_F(f)\|_1.
\end{aligned}$$

3 Estimates of flag Poisson integrals

In this section, we will show the following estimates:

$$\|S_F(u)\|_1 \lesssim \|u^*\|_1 \lesssim \|u^+\|_1 \lesssim \sum_{j=1}^{n+m} \sum_{k=1}^m \|R_j^{(1)} R_k^{(2)}(f)\|_1 + \|f\|_1.$$

3.1 The estimate $\|S_F(u)\|_1 \lesssim \|u^*\|_1$

We first introduce the following maximal function associated with the flag structure.

Definition 3.1. For $f \in L^1_{loc}(\mathbb{R}^{n+m})$, we define the maximal function by

$$M_F(f)(x, y) = \sup_{t,s>0, (x,y) \in R} \frac{1}{|R|} \int_R |f(u, v)| du dv,$$

where $R = I \times J$ run over all rectangles with sides parallel to the axes and $\ell(I) = t$, $\ell(J) = t + s$.

We now recall the Lemma of K. Merryfield.

Lemma 3.2 ([19]). Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfy

- (1) $\varphi(-x) = \varphi(x)$;
- (2) $\text{supp } \varphi \subset B_n(0, 1)$, where $B_n(0, 1)$ is the unit ball in \mathbb{R}^n ;
- (3) $\int_{\mathbb{R}^n} \varphi(x) dx = 1$.

Then there exists a function $\psi \in C_0^\infty(\mathbb{R}^n)$ that satisfies $\text{supp } \psi \subset B_n(0, 1)$ and $\int_{\mathbb{R}^n} \psi(x) dx = 0$, such that for $u(x, t) = P_t * f(x)$ we have

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}} |\nabla u(x, t)|^2 |g * \varphi_t(x)|^2 t dx dt \\
&\leq C \int_{\mathbb{R}^n} f^2(x) g^2(x) dx + \int_{\mathbb{R}_+^{n+1}} u^2(x, t) |g * \psi_t(x)|^2 \frac{dx dt}{t},
\end{aligned}$$

where C is independent of f and g .

Now we establish a K. Merryfield type lemma in this flag setting as follows. Let $\varphi^{(1)}(x, y) \in C_0^\infty(\mathbb{R}^{n+m})$ satisfy

- (1) $\varphi^{(1)}(-x, -y) = \varphi^{(1)}(x, y)$;
- (2) $\text{supp } \varphi^{(1)} \subset B_{n+m}(0, 1)$, where $B_{n+m}(0, 1)$ is the unit ball in \mathbb{R}^{n+m} ;
- (3) $\int_{\mathbb{R}^{n+m}} \varphi(x, y) dx dy = 1$.

Let $\varphi^{(2)}(z) \in C_0^\infty(\mathbb{R}^m)$ satisfies the same conditions as in Lemma 3.2, and $\varphi(x, y) = \varphi^{(1)} *_{\mathbb{R}^m} \varphi^{(2)}(x, y)$.

Similarly, we can obtain two functions $\psi^{(1)}(x, y)$ and $\psi^{(2)}(z)$ such that $\psi^{(1)} \in C_0^\infty(\mathbb{R}^{n+m})$ that satisfies $\text{supp } \psi^{(1)} \subset B_{n+m}(0, 1)$ and

$$\int_{\mathbb{R}^{n+m}} \psi^{(1)}(x, y) dx = 0,$$

and $\psi^{(2)} \in C_0^\infty(\mathbb{R}^m)$ that satisfies $\text{supp } \psi^{(2)} \subset B_m(0, 1)$ and

$$\int_{\mathbb{R}^m} \psi^{(2)}(z) dz = 0.$$

Then we define $\psi(x, y) := \psi^{(1)} *_{\mathbb{R}^m} \psi^{(2)}(x, y)$. We arrive at the following technical lemma.

Lemma 3.3. *Let all the notation be the same as above.*

$$\begin{aligned} & \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{m+1}} |t \nabla^{(1)} s \nabla^{(2)} P_{t,s} * f(x, y)|^2 |g * \varphi_{t,s}(x, y)|^2 \frac{dy ds}{s} \frac{dx dt}{t} \\ & \leq C \left\{ \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f^2(x, y) g^2(x, y) dx dy \right. \\ & \quad + \int_{\mathbb{R}^n} \int_{\mathbb{R}_+^{m+1}} |P_s^{(2)} *_{\mathbb{R}^m} f(x, y)|^2 |\psi_s^{(2)} *_{\mathbb{R}^m} g(x, y)|^2 \frac{dy ds}{s} dx \\ & \quad + \int_{\mathbb{R}^m} \int_{\mathbb{R}_+^{n+1}} |P_t^{(1)} * f(x, y)|^2 |\psi_t^{(1)} * g(x, y)|^2 \frac{dx dt}{t} dy \\ & \quad \left. + \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{m+1}} |P_{t,s} * f(x, y)|^2 |\psi_{t,s} * g(x, y)|^2 \frac{dy ds}{s} \frac{dx dt}{t} \right\}. \end{aligned} \quad (3.1)$$

Proof. Applying Lemma 3.2 with replacing n by $n + m$ gives

$$\begin{aligned} & \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{m+1}} |t \nabla^{(1)} s \nabla^{(2)} u(x, y, t, s)|^2 |g * \varphi_{t,s}(x, y)|^2 \frac{dy ds}{s} \frac{dx dt}{t} \\ & = \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |t \nabla^{(1)} P_t^{(1)} * ((s \nabla^{(2)} P_s^{(2)}) *_2 f)(x, y)|^2 |\varphi_t^{(1)} * (\varphi_s^{(2)} *_2 g)(x, y)|^2 dx dy \frac{dt}{t} \frac{ds}{s} \\ & \leq C \int_0^\infty \int_{\mathbb{R}^n \times \mathbb{R}^m} |F_s(x, y)|^2 |G_s(x, y)|^2 dx dy \frac{ds}{s} \end{aligned}$$

$$\begin{aligned}
& + \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n \times \mathbb{R}^m} |P_t^{(1)} * F_s(x, y)|^2 |\tilde{Q}_t^{(1)} * G_s(x, y)|^2 dx dy \frac{dt}{t} \frac{ds}{s} \\
& \triangleq I_1 + I_2,
\end{aligned}$$

where $F_s(x, y) = (s\nabla^{(2)} P_s^{(2)}) *_{\mathbb{R}^m} f(x, y)$ and $G_s(x, y) = \varphi_s^{(2)} *_{\mathbb{R}^m} g(x, y)$.

To estimate I_1 , we have

$$\begin{aligned}
& \int_0^\infty \int_{\mathbb{R}^n \times \mathbb{R}^m} |F_s(x, y)|^2 |G_s(x, y)|^2 dx dy \frac{ds}{s} \\
& = \int_{\mathbb{R}^n} \int_{\mathbb{R}_+^{m+1}} \left| \left(s\nabla^{(2)} P_s^{(2)} *_{\mathbb{R}^m} f(x, \cdot) \right) (y) \right|^2 \left| \left(\varphi_s^{(2)} *_{\mathbb{R}^m} g(x, \cdot) \right) (y) \right|^2 \frac{dy ds}{s} dx \\
& \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f^2(x, y) g^2(x, y) dx dy \\
& \quad + C \int_{\mathbb{R}^n} \int_{\mathbb{R}_+^{m+1}} \left| P_s^{(2)} *_{\mathbb{R}^m} f(x, y) \right|^2 \left| \tilde{Q}_s^{(2)} *_{\mathbb{R}^m} g(x, y) \right|^2 \frac{dy ds}{s} dx,
\end{aligned}$$

where the last inequality follows again from Lemma 3.2.

Similarly,

$$\begin{aligned}
I_2 & = \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{m+1}} \left| s\nabla^{(2)} P_s^{(2)} *_{\mathbb{R}^m} (P_t^{(1)} * f(x, \cdot)) (y) \right|^2 \left| \varphi_s^{(2)} *_{\mathbb{R}^m} (\tilde{Q}_t^{(1)} * g(x, \cdot)) (y) \right|^2 \frac{dy ds}{s} \frac{dx dt}{t} \\
& \leq C \int_{\mathbb{R}^m} \int_{\mathbb{R}_+^{n+1}} |P_t^{(1)} * f^2(x, y)|^2 |\tilde{Q}_t^{(1)} * g^2(x, y)|^2 \frac{dx dt}{t} dy \\
& \quad + C \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{m+1}} |P_s^{(2)} *_{\mathbb{R}^m} P_t^{(1)} * f(x, y)|^2 |\tilde{Q}_s^{(2)} *_{\mathbb{R}^m} \tilde{Q}_t^{(1)} * g(x, y)|^2 \frac{dy ds}{s} \frac{dx dt}{t}.
\end{aligned}$$

The estimates of term I_1 and term I_2 yield (3.1). \square

We now begin to prove $\|S_F(u)\|_1 \lesssim \|u^*\|_1$. For any $\alpha > 0$ and each $f \in L^1(\mathbb{R}^{n+m})$ satisfying $\|u^*\|_1 < \infty$, define

$$A(\alpha) = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m : M_F(\chi_{\{u^* > \alpha\}})(x, y) < \frac{1}{200} \right\}.$$

Then we have

$$\begin{aligned}
& \int_{A(\alpha)} S_F^2(u)(x, y) dx dy \\
& \leq \int_{\mathbb{R}_+^{n+1} \times \mathbb{R}_+^{m+1}} \int_{A(\alpha)} \chi_{t,s}(x - x_1, y - y_1) dx dy |t\nabla^{(1)} s\nabla^{(2)} u(x_1, y_1, t, s)|^2 dx_1 dy_1 \frac{dt}{t} \frac{ds}{s}.
\end{aligned}$$

By the definition of $\chi_{t,s}(x - x_1, y - y_1)$, for any fixed (x_1, y_1, t, s) , if $\chi_{t,s}(x - x_1, y - y_1) \neq 0$, then (x, y) belongs to R , where $R = R(x_1, y_1, t, s)$ is a rectangle centered at (x_1, y_1) and with side-length $2t$ and $2t + 2s$. This means that to estimate $\int_{A(\alpha)} \chi_{t,s}(x - x_1, y - y_1) dx dy$, we only need to consider those $(x, y) \in A(\alpha) \cap R(x_1, y_1, t, s)$. As a consequence,

$$M_F(\chi_{\{u^* > \alpha\}})(x, y) < \frac{1}{200}.$$

Hence for such fixed (x_1, y_1, t, s) mentioned above, we have

$$\frac{1}{|R(x_1, y_1, t, s)|} |A(\alpha) \cap R(x_1, y_1, t, s)| < \frac{1}{200}.$$

Let $R^* = \left\{ (x_1, y_1, t, s) : \frac{1}{|R(x_1, y_1, t, s)|} |A(\alpha) \cap R(x_1, y_1, t, s)| < \frac{1}{200} \right\}$, then we have

$$\int_{A(\alpha)} S_F^2(u)(x, y) dx dy \leq \int_{R^*} |t \nabla^{(1)} s \nabla^{(2)} u(x_1, y_1, t, s)|^2 dx_1 dy_1 \frac{dt}{t} \frac{ds}{s}. \quad (3.2)$$

Let $g(x, y) = \chi_{\{u^* \leq \alpha\}}(x, y)$ and $\varphi^{(1)}(x, y) \in C_0^\infty(\mathbb{R}^{n+m})$ satisfy

- (1) $\varphi^{(1)}(-x, -y) = \varphi^{(1)}(x, y)$;
- (2) $\text{supp } \varphi^{(1)} \subset B_{n+m}(0, 1)$, where $B_{n+m}(0, 1)$ is the unit ball in \mathbb{R}^{n+m} ;
- (3) $\int_{\mathbb{R}^{n+m}} \varphi(x, y) dx dy = 1$;
- (4) $\varphi^{(1)}(x, y) = 1$ when $|(x, y)| \leq \frac{1}{3}$.

Similarly, $\varphi^{(2)}(x, y) \in C_0^\infty(\mathbb{R}^m)$ satisfies

- (1) $\varphi^{(2)}(-z) = \varphi^{(2)}(z)$;
- (2) $\text{supp } \varphi^{(2)} \subset B_m(0, 1)$, where $B_m(0, 1)$ is the unit ball in \mathbb{R}^m ;
- (3) $\int_{\mathbb{R}^m} \varphi^{(2)}(z) dz = 1$;
- (4) $\varphi^{(2)}(z) = 1$ when $|z| \leq \frac{1}{3}$.

Set $\varphi(x, y) = \varphi^{(1)} *_{\mathbb{R}^m} \varphi^{(2)}(x, y)$. Then for $(x_1, y_1) \in R^*$, we have

$$\begin{aligned} \varphi_{t,s} * g(x_1, y_1) &= \int_{\{u^* \leq \alpha\}} \varphi_{t,s}(x_1 - x_1, y - y_1) dx_1 dy_1 \\ &\geq \int_{\{u^* \leq \alpha\} \cap R(x, y, t, s)} \varphi_{t,s}(x - x_1, y - y_1) dx_1 dy_1 \\ &\geq C, \end{aligned} \quad (3.3)$$

where the last inequality follows from the definition of R^* . Combining (3.2) and (3.3), we have

$$\begin{aligned} &\int_{A(\alpha)} S^2(u)(x, y) dx dy \\ &\leq C \int_{R^*} |t \nabla^{(1)} s \nabla^{(2)} u(x_1, y_1, t, s)|^2 |\varphi_{t,s} * g(x_1, y_1)|^2 dx_1 dy_1 \frac{dt}{t} \frac{ds}{s} \\ &\leq C \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{m+1}} |t \nabla^{(1)} s \nabla^{(2)} u(x, y, t, s)|^2 |g * \varphi_{t,s}(x, y)|^2 \frac{dy ds}{s} \frac{dx dt}{t} \\ &\leq C \left\{ \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f^2(x, y) g^2(x, y) dx dy \right. \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R}^n} \int_{\mathbb{R}_+^{m+1}} |P_s^{(2)} *_{\mathbb{R}^m} f(x, y)|^2 |\psi_s^{(2)} *_{\mathbb{R}^m} g(x, y)|^2 \frac{dy ds}{s} dx \\
& + \int_{\mathbb{R}^m} \int_{\mathbb{R}_+^{n+1}} |P_t^{(1)} * f(x, y)|^2 |\psi_t^{(1)} * g(x, y)|^2 \frac{dx dt}{t} dy \\
& + \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{m+1}} |P_{t,s} * f(x, y)|^2 |\psi_{t,s} * g(x, y)|^2 \frac{dy ds}{s} \frac{dx dt}{t} \Big\} \\
& =: II_1 + II_2 + II_3 + II_4,
\end{aligned}$$

where the last inequality follows from Lemma 3.3.

For the term II_1 , we have

$$\begin{aligned}
|II_1| & \leq C \int_{\mathbb{R}^m} \int_{\mathbb{R}^{n+1}} |P_t^{(1)} * f(x, y)|^2 |\psi_t^{(1)} * g(x, y)|^2 \frac{dx dt}{t} dy \\
& \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f^2(x, y) g^2(x, y) dx dy \\
& \leq C \int_{\{u^* \leq \alpha\}} f^2(x, y) dx dy \\
& \leq C \int_{\{u^* \leq \alpha\}} |u^*(x, y)|^2 dx dy.
\end{aligned}$$

If $\psi_s^{(2)} *_{\mathbb{R}^m} g(x, y) = \int \psi_s^{(2)}(y - w) g(x, w) dw \neq 0$, then there exists some w such that $|y - w| < s$ and $(x, w) \in \{u^* \leq \alpha\}$. Hence $|P_s^{(2)} *_{\mathbb{R}^m} f(x, y)| \leq \alpha$. As a consequence,

$$\begin{aligned}
|II_2| & \leq C \alpha^2 \int_{\mathbb{R}^n} \int_{\mathbb{R}_+^{m+1}} |\psi_s^{(2)} *_{\mathbb{R}^m} g(x, y)|^2 \frac{dy ds}{s} dx \\
& = C \alpha^2 \int_{\mathbb{R}^n} \int_{\mathbb{R}_+^{m+1}} |\psi_s^{(2)} *_{\mathbb{R}^m} (1 - g(x, y))|^2 \frac{dy ds}{s} dx \\
& \lesssim \alpha^2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |1 - g(x, y)|^2 dy dx \\
& \lesssim \alpha^2 |\{u^* > \alpha\}|.
\end{aligned} \tag{3.4}$$

The proof of the estimate for II_3 is similar to II_2 .

For the last term II_4 , if $\psi_{t,s} * g(x, y) = \int \psi_{t,s}(x - v, y - w) g(v, w) dv dw \neq 0$, similarly as term II_2 , there exists (v, w) such that $(v, w) \in \{u^* \leq \alpha\}$ and $|x - v| < t$, $|y - w| < t + s$. Hence $|P_{t,s} * f(x, y)| \leq \alpha$. Following the same routine of (3.4), we have

$$|II_4| \leq C \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{m+1}} |P_{t,s} * f(x, y)|^2 |\psi_{t,s} * g(x, y)|^2 \frac{dy ds}{s} \frac{dx dt}{t} \Big\} \leq C \alpha^2 |\{u^* > \alpha\}|.$$

Combining all estimates above implies that

$$\int_{\{M_F(\chi_{\{u^* > \alpha\}}) \leq \frac{1}{200}\}} S_F^2(u)(x, y) dx dy \leq C \left(\alpha^2 |\{u^* > \alpha\}| + \int_{\{u^* \leq \alpha\}} |u^*(x, y)|^2 dx dy \right). \tag{3.5}$$

By the definition of the maximal function M_F , we have

$$\begin{aligned}
\left| \left\{ (x, y) : M_F(\chi_{\{u^* > \alpha\}}) > \frac{1}{200} \right\} \right| &\leq C \left| \left\{ (x, y) : M_s(\chi_{\{u^* > \alpha\}})(x, y) > \frac{1}{200} \right\} \right| \\
&\leq C \int_{\mathbb{R}^{n+m}} M_s(\chi_{\{u^* > \alpha\}})^2(x, y) dx dy \\
&\leq C \int_{\mathbb{R}^{n+m}} \chi_{\{u^* > \alpha\}}^2(x, y) dx dy \\
&\leq C |\{u^* > \alpha\}|.
\end{aligned} \tag{3.6}$$

Inserting (3.6) into (3.5) yields

$$\begin{aligned}
|\{(x, y) : S_F(u)(x, y) > \alpha\}| &\leq \left| \left\{ (x, y) : M_F(\chi_{\{u^* > \alpha\}}) \leq \frac{1}{200} \text{ and } S_F(u)(x, y) > \alpha \right\} \right| \\
&\quad + \left| \left\{ (x, y) : M_F(\chi_{\{u^* > \alpha\}}) > \frac{1}{200} \text{ and } S_F(u)(x, y) > \alpha \right\} \right| \\
&\leq C \left(\alpha^2 |\{u^* > \alpha\}| + \int_{\{u^* \leq \alpha\}} |u^*(x, y)|^2 dx dy \right),
\end{aligned}$$

which implies that $\|S_F(u)\|_1 \leq C \|u^*\|_1$.

3.2 The estimate $\|u^*\|_1 \lesssim \|u^+\|_1$

As mentioned in the introduction, the flag Hardy space is in some sense intermediate between the classical one parameter and the product Hardy spaces. To deal with the flag non-tangential maximal function, we decompose it by the classical one parameter and the product cases. More precisely, we write, for any $(\bar{x}, \bar{y}) \in \mathbb{R}^{n+m}$, that

$$\begin{aligned}
u^*(\bar{x}, \bar{y}) &= \sup_{(x, y, t, s) \in \Gamma(\bar{x}, \bar{y})} |u(x, y, t, s)| \\
&\leq \sup_{(x, y, t, s) \in \Gamma(\bar{x}, \bar{y}), s \leq t} |u(x, y, t, s)| + \sup_{(x, y, t, s) \in \Gamma(\bar{x}, \bar{y}), s > t} |u(x, y, t, s)| \\
&=: u_1^*(\bar{x}, \bar{y}) + u_2^*(\bar{x}, \bar{y}),
\end{aligned}$$

where $\Gamma(\bar{x}, \bar{y}) = \{(x, y, t, s) : |x - \bar{x}| \leq t, |y - \bar{y}| \leq t + s\}$.

The main idea to show $\|u_1^*\|_1 \lesssim \|u^+\|_1$ is the following lemma which was proved by Fefferman and Stein in [4] for the classical one parameter Hardy space.

Lemma 3.4. *Suppose B is a ball in \mathbb{R}^{d+1} , with center (x_0, t_0) . Let u be harmonic in B and continuous on the closure of B . For any $p > 0$,*

$$|u(x_0, t_0)|^p \leq C_p \frac{1}{|B|} \int_B |u(x, t)|^p dx dt.$$

Suppose $f \in L^1(\mathbb{R}^{n+m})$ and $u(x, y, t, s) = P_{t,s} * f(x, y)$. Note that $u(x, y, t, s)$, as a function of (x, y, t) with a fixed s , is harmonic on \mathbb{R}_+^{n+m+1} . Lemma 3.4 implies that for any $r > 0$ and

$s \leq t$,

$$|u(x, y, t, s)|^r \leq C_r \frac{1}{|B_1|} \int_{B_1} |u(x_1, y_1, t_1, s)|^r dx_1 dy_1 dt_1,$$

where B_1 is any ball in \mathbb{R}_+^{n+m+1} with the radius t and the center $(x, y, t) \in \Gamma_1(\bar{x}, \bar{y})$, where

$$\Gamma_1(\bar{x}, \bar{y}) = \{(x_1, y_1, t) : |\bar{x} - x_1| \leq 2t, |\bar{y} - y_1| \leq 2t\}.$$

Note that the projection of B_1 on \mathbb{R}^{n+m} is contained in the ball centered at (\bar{x}, \bar{y}) with radius $4t$. Therefore,

$$\begin{aligned} |u(x, y, t, s)|^r &\leq C_r t^{-n-m} \int_{B((\bar{x}, \bar{y}), 4t)} |u(x_1, y_1, t_1, s)|^r dx_1 dy_1 \\ &\leq C_r t^{-n-m} \int_{B((\bar{x}, \bar{y}), 4t)} |u^+(x_1, y_1)|^r dx_1 dy_1 \\ &\leq C_r M_1(|u^+|^r)(\bar{x}, \bar{y}), \end{aligned}$$

where M_1 is the standard Hardy–Littlewood maximal function on \mathbb{R}^{n+m} .

As a consequence, this implies that

$$u_1^*(\bar{x}, \bar{y}) \leq C \left(M_1(|u^+|^r)(\bar{x}, \bar{y}) \right)^{\frac{1}{r}},$$

which, together with the $L^{\frac{1}{r}}, 0 < r < 1$, boundedness of the Hardy–Littlewood maximal function $M_1(f)$, implies that

$$\|u_1^*\|_1 \leq C \|u^+\|_1.$$

Now we estimate $u_2^*(\bar{x}, \bar{y})$. Observe that when $s > t$ the cone $\Gamma(\bar{x}, \bar{y}) = \{(x_1, y_1, t) : |\bar{x} - x_1| \leq t, |\bar{y} - y_1| \leq t + s\}$ essentially is the cone in the product setting. Therefore, we write that

$$\begin{aligned} u_2^*(\bar{x}, \bar{y}) &= \sup_{(x, y, t, s) \in \Gamma(\bar{x}, \bar{y}), s > t} |P_{t, s} * f(x, y)| \\ &\leq \sup_{(x, y, t, s) \in \Gamma_2(\bar{x}, \bar{y})} \left| \int_{\mathbb{R}^n \times \mathbb{R}^m} \int_{\mathbb{R}^m} P_t^{(1)}(x - x_1, z) P_s^{(2)}(y - y_1 - z) dz f(x_1, y_1) dx_1 dy_1 \right|, \end{aligned}$$

where

$$\Gamma_2(\bar{x}, \bar{y}) = \{(x, y, t, s) : |\bar{x} - x| \leq 2t, |\bar{y} - y| \leq 2s\}.$$

The main idea to estimate the last term above is to introduce the following flag grand maximal function $\mathcal{G}_{\beta, \gamma}(f)(x_0, y_0)$: for $f \in L^1(\mathbb{R}^{n+m})$ and $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$\mathcal{G}_{\beta, \gamma}(f)(x_0, y_0) := \sup\{|\langle f, \varphi \rangle| : \|\varphi\|_{\widetilde{\mathcal{M}}_{flag}(\beta, \gamma, r_1, r_2, x_0, y_0)} \leq 1, r_1, r_2 > 0\}.$$

By Definition 2.4, it is easy to see that as a function of (x_1, y_1) ,

$$\int_{\mathbb{R}^m} P_t^{(1)}(x - x_1, z) P_s^{(2)}(y - y_1 - z) dz$$

is in $\widetilde{\mathcal{M}}_{flag}(1, 1, t, s, \bar{x}, \bar{y})$ with $(x, y, t, s) \in \Gamma_2(\bar{x}, \bar{y})$ since $P_t^{(1)}(x - x_1, z) \in \widetilde{\mathcal{M}}_{n+m}(1, 1, t, \bar{x}, 0)$ and $P_s^{(2)}(y - y_1) \in \widetilde{\mathcal{M}}_m(1, 1, s, \bar{y})$. Moreover, it is also easy to check that

$$\sup_{(x, y, t, s) \in \Gamma_2(\bar{x}, \bar{y})} \left\| \int_{\mathbb{R}^m} P_t^{(1)}(x - x_1, z) P_s^{(2)}(y - y_1 - z) dz \right\|_{\widetilde{\mathcal{M}}_{flag}(1, 1, t, s, \bar{x}, \bar{y})} \leq C,$$

where C is an absolute constant independent of (\bar{x}, \bar{y}) .

As a consequence, we obtain that

$$\begin{aligned} u_2^*(\bar{x}, \bar{y}) &= \sup_{(x, y, t, s) \in \Gamma_2(\bar{x}, \bar{y})} \left| \left\langle \int_{\mathbb{R}^m} P_t^{(1)}(x - \cdot, z) P_s^{(2)}(y - \cdot - z) dz, f(\cdot, \cdot) \right\rangle \right| \\ &\leq C \mathcal{G}_{1,1}(f)(\bar{x}, \bar{y}). \end{aligned}$$

It suffices to prove that for $f \in L^1(\mathbb{R}^{n+m})$ and $r > 0$,

$$\mathcal{G}_{1,1}(f)(\bar{x}, \bar{y}) \leq C \left(M_1 \left(M_2(|u^+|^r) \right) (\bar{x}, \bar{y}) \right)^{\frac{1}{r}} + C \left(M_2 \left(M_1(|u^+|^r) \right) (\bar{x}, \bar{y}) \right)^{\frac{1}{r}}, \quad (3.7)$$

where M_1 and M_2 are the Hardy-Littlewood maximal functions on \mathbb{R}^{n+m} and \mathbb{R}^m , respectively.

We first claim that

$$|\langle f, \psi \rangle| \leq C \left(M_1 \left(M_2(|u^+|^r) \right) (\bar{x}, \bar{y}) \right)^{\frac{1}{r}} \quad (3.8)$$

for $r < 1$ and close to 1, $f \in L^1(\mathbb{R}^{n+m})$, and for every $\psi \in \mathcal{M}_{flag}(1, 1, 2^{-j_1}, 2^{-k_1}, \bar{x}, \bar{y})$ with the norm $\|\psi\|_{\mathcal{M}_{flag}(1, 1, 2^{-j_1}, 2^{-k_1}, \bar{x}, \bar{y})} \leq 1$.

The key idea to show the above claim is to apply the discrete Calderón reproducing formula. To see this, consider the following approximations to the identity on \mathbb{R}^{n+m} : For each $j \in \mathbb{Z}$, define the operator

$$\mathcal{P}_j^{(1)} := P_{2^{-j}}^{(1)}$$

with the kernel $\mathcal{P}_j^{(1)}(x, y) := P_{2^{-j}}^{(1)}(x, y)$.

It is easy to see that

$$\lim_{j \rightarrow \infty} \mathcal{P}_j^{(1)} = \lim_{j \rightarrow \infty} P_{2^{-j}}^{(1)} = Id \quad \text{and} \quad \lim_{j \rightarrow -\infty} \mathcal{P}_j^{(1)} = \lim_{j \rightarrow -\infty} P_{2^{-j}}^{(1)} = 0$$

in the sense of $L^2(\mathbb{R}^{n+m})$. And we further have

$$\int_{\mathbb{R}^{n+m}} \mathcal{P}_j^{(1)}(x, y) dx dy = 1.$$

Set $Q_j^{(1)} := \mathcal{P}_j^{(1)} - \mathcal{P}_{j-1}^{(1)}$. Then $Q_j^{(1)}(x, y)$, the kernel of $Q_j^{(1)}$ satisfies the same size and smoothness conditions as $\mathcal{P}_j^{(1)}(x, y)$ does, and

$$\int_{\mathbb{R}^{n+m}} Q_j^{(1)}(x, y) dx dy = 0.$$

The operators $\mathcal{P}_k^{(2)}$ and $Q_k^{(2)}$ on \mathbb{R}^m are defined similarly.

Repeating the same proof as in Theorem 2.5, we have the following reproducing formula: there exist functions $\phi_{j,k}(x, y, x_I, y_J) \in \mathcal{M}_{flag}(\beta, \gamma, 2^{-j}, 2^{-k}, x_I, y_J)$ and a fixed large integer N such that for $f(x, y) = \int_{\mathbb{R}^m} f_1(x, y - z) f_2(z) dz$ with $f_1 \in \mathcal{M}_{n+m}(\beta, \gamma, r_1, x_0, y_0)$ and $f_2 \in \mathcal{M}_m(\beta, \gamma, r_2, z_0)$,

$$f(x, y) = \sum_j \sum_k \sum_I \sum_J |I||J| \phi_{j,k}(x, y, x_I, y_J) Q_{j,k}(f)(x_I, y_J), \quad (3.9)$$

where the series converges in $L^2(\mathbb{R}^{n+m})$ and in the flag test function space, and $I \subset \mathbb{R}^n, J \subset \mathbb{R}^m$ are dyadic cubes with side-lengths $\ell(I) = 2^{-j-N}, \ell(J) = 2^{-(j \wedge k - N)}, x_I$ and y_J are any fixed points in I and J , respectively, and

$$Q_{j,k}(f)(x_I, y_J) = \int_{\mathbb{R}^{n+m}} Q_{j,k}(x_I - x, y_J - y) f(x, y) dx dy$$

with the kernel

$$Q_{j,k}(x, y) = \int_{\mathbb{R}^m} Q_j^{(1)}(x, y - z) Q_k^{(2)}(z) dz.$$

Now applying (3.9) to the left-hand side of (3.8), we have

$$\begin{aligned} |\langle f, \psi \rangle| &= \left| \sum_j \sum_k \sum_I \sum_J |I||J| \langle \psi, \phi_{j,k}(\cdot, \cdot, x_I, y_J) \rangle Q_{j,k}(f)(x_I, y_J) \right| \\ &\leq C \sum_j \sum_k \sum_I \sum_J |I||J| 2^{-|j-j_1|\beta} 2^{-|k-k_1|\beta} \frac{2^{-(j \wedge j_1)\gamma}}{(2^{-j \wedge j_1} + |x_I - \bar{x}|)^{n+\gamma}} \\ &\quad \times \frac{2^{-[(k \wedge k_1) \wedge (j \wedge j_1)]\gamma}}{(2^{-[(k \wedge k_1) \wedge (j \wedge j_1)]} + |y_J - \bar{y}|)^{m+\gamma}} \inf_{z_1 \in I, z_2 \in J} |u^+(z_1, z_2)|. \end{aligned} \quad (3.10)$$

Here in the last inequality we use the following estimates:

(1) the almost orthogonality estimate:

$$\begin{aligned} &|\langle \psi, \phi_{j,k}(\cdot, \cdot, x_I, y_J) \rangle| \\ &\leq C 2^{-|j-j_1|\beta} 2^{-|k-k_1|\beta} \frac{2^{-(j \wedge j_1)\gamma}}{(2^{-j \wedge j_1} + |x_I - \bar{x}|)^{n+\gamma}} \frac{2^{-[(k \wedge k_1) \wedge (j \wedge j_1)]\gamma}}{(2^{-[(k \wedge k_1) \wedge (j \wedge j_1)]} + |y_J - \bar{y}|)^{m+\gamma}} \end{aligned}$$

for $\beta, \gamma < 1$. For the proof see [16, page 2840] for the one-parameter case and [24] for similar estimates on homogeneous groups.

- (2) the fact that x_I and y_J are any fixed points in I and J , implies that we can choose $x_I \in I$ and $y_J \in J$ such that

$$\begin{aligned}
& |Q_{j,k}(f)(x_I, y_J)| \\
& \leq 2 \inf_{z_1 \in I, z_2 \in J} |Q_{j,k}(f)(z_1, z_2)| \\
& = 2 \inf_{z_1 \in I, z_2 \in J} \left| \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^m} Q_j^{(1)}(z_1 - x, z_2 - y - z) Q_k^{(2)}(z) dz f(x, y) dx dy \right| \\
& = 2 \inf_{z_1 \in I, z_2 \in J} \left| \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^m} \left(P_j^{(1)}(z_1 - x, z_2 - y - z) - P_{j-1}^{(1)}(z_1 - x, z_2 - y - z) \right) \right. \\
& \quad \times \left. \left(P_k^{(2)}(z) - P_{k-1}^{(2)}(z) \right) dz f(x, y) dx dy \right| \\
& \leq 8 \inf_{z_1 \in I, z_2 \in J} |u^+(z_1, z_2)|.
\end{aligned}$$

To estimate the last term in (3.10), observe that for $0 < r < 1$,

$$\begin{aligned}
& \sum_j \sum_k \sum_I \sum_J |I||J| 2^{-|j-j_1|\beta} 2^{-|k-k_1|\beta} \frac{2^{-(j \wedge j_1)\gamma}}{(2^{-j \wedge j_1} + |x_I - \bar{x}|)^{n+\gamma}} \frac{2^{-[(k \wedge k_1) \wedge (j \wedge j_1)]\gamma}}{(2^{-[(k \wedge k_1) \wedge (j \wedge j_1)]} + |y_J - \bar{y}|)^{m+\gamma}} \\
& \quad \times \inf_{z_1 \in I, z_2 \in J} |u^+(z_1, z_2)| \\
& \leq \left\{ \sum_j \sum_k \sum_I \sum_J |I|^r |J|^r 2^{-|j-j_1|\beta r} 2^{-|k-k_1|\beta r} \frac{2^{-(j \wedge j_1)\gamma r}}{(2^{-j \wedge j_1} + |x_I - \bar{x}|)^{(n+\gamma)r}} \right. \\
& \quad \times \left. \frac{2^{-[(k \wedge k_1) \wedge (j \wedge j_1)]\gamma r}}{(2^{-[(k \wedge k_1) \wedge (j \wedge j_1)]} + |y_J - \bar{y}|)^{(m+\gamma)r}} \inf_{z_1 \in I, z_2 \in J} |u^+(z_1, z_2)|^r \right\}^{1/r}.
\end{aligned}$$

Note that $\ell(I) = 2^{-j-N}$ and $\ell(J) = 2^{-(j \wedge k-N)}$. Write

$$\begin{aligned}
& \sum_I \sum_J |I|^r |J|^r \frac{2^{-(j \wedge j_1)\gamma r}}{(2^{-j \wedge j_1} + |x_I - \bar{x}|)^{(n+\gamma)r}} \frac{2^{-[(k \wedge k_1) \wedge (j \wedge j_1)]\gamma r}}{(2^{-[(k \wedge k_1) \wedge (j \wedge j_1)]} + |y_J - \bar{y}|)^{(m+\gamma)r}} \inf_{z_1 \in I, z_2 \in J} |u^+(z_1, z_2)|^r \\
& = C 2^{-jn(r-1)} 2^{(j \wedge k)m(r-1)} \sum_I \sum_J |I||J| \frac{2^{-(j \wedge j_1)\gamma r}}{(2^{-j \wedge j_1} + |x_I - \bar{x}|)^{(n+\gamma)r}} \frac{2^{-[(k \wedge k_1) \wedge (j \wedge j_1)]\gamma r}}{(2^{-[(k \wedge k_1) \wedge (j \wedge j_1)]} + |y_J - \bar{y}|)^{(m+\gamma)r}} \\
& \quad \times \inf_{z_1 \in I, z_2 \in J} |u^+(z_1, z_2)|^r \\
& \leq C 2^{-jn(r-1)} 2^{(j \wedge k)m(r-1)} \int_{\mathbb{R}^n \times \mathbb{R}^m} \frac{2^{-(j \wedge j_1)\gamma r}}{(2^{-j \wedge j_1} + |x - \bar{x}|)^{(n+\gamma)r}} \frac{2^{-[(k \wedge k_1) \wedge (j \wedge j_1)]\gamma r}}{(2^{-[(k \wedge k_1) \wedge (j \wedge j_1)]} + |y - \bar{y}|)^{(m+\gamma)r}} \\
& \quad \times |u^+(x, y)|^r dx dy \\
& \leq C 2^{-jn(r-1)} 2^{(j \wedge k)m(r-1)} 2^{-(j \wedge j_1)\gamma r} 2^{-(j \wedge j_1)[n-(n+\gamma)r]} 2^{-[(k \wedge k_1) \wedge (j \wedge j_1)]\gamma r} \\
& \quad \times 2^{-[(k \wedge k_1) \wedge (j \wedge j_1)][m-(m+\gamma)r]} \left(M_1(M_2(|u^+|^r)) \right)(\bar{x}, \bar{y}).
\end{aligned}$$

A simple computation shows that if $\frac{m+n}{m+n+\beta} < r < 1$, then

$$\sum_j \sum_k 2^{-|j-j_1|\beta r} 2^{-|k-k_1|\beta r} 2^{-jn(r-1)} 2^{-(j \wedge j_1)(n-nr)} 2^{(j \wedge k)m(r-1)} 2^{-[(k \wedge k_1) \wedge (j \wedge j_1)][m-mr]} \leq C.$$

Thus, we obtain that the right-hand side of (3.10) is bounded by

$$\left(M_1 \left(M_2(|u^+|^r) \right) (\bar{x}, \bar{y}) \right)^{\frac{1}{r}},$$

which implies (3.8).

We now prove (3.7). For every φ with

$$\varphi(x, y) = \int_{\mathbb{R}^m} \varphi^{(1)}(x, y - z) \varphi^{(2)}(z) dz,$$

where $\varphi^{(1)}(x, y) \in \widetilde{\mathcal{M}}_{n+m}(1, 1, t, \bar{x}, 0)$ with $\|\varphi^{(1)}\|_{\widetilde{\mathcal{M}}_{n+m}(1, 1, t, \bar{x}, 0)} \leq 1$, and $\varphi^{(2)}(z) \in \widetilde{\mathcal{M}}_m(1, 1, s, \bar{y})$ with $\|\varphi^{(2)}\|_{\widetilde{\mathcal{M}}_m(1, 1, s, \bar{y})} \leq 1$, we have $\varphi(x, y) \in \widetilde{\mathcal{M}}_{flag}(1, 1, t, s, \bar{x}, \bar{y})$ with $\|\varphi\|_{\widetilde{\mathcal{M}}_{flag}(1, 1, t, s, \bar{x}, \bar{y})} \leq 1$.

Let

$$\sigma_1 := \int_{\mathbb{R}^{n+m}} \varphi^{(1)}(x, y) dx dy, \quad \sigma_2 := \int_{\mathbb{R}^m} \varphi^{(2)}(z) dz.$$

It is obvious that $|\sigma_1|, |\sigma_2| \leq C$. We set

$$\begin{aligned} \psi^{(1)}(x, y) &:= \frac{1}{1 + \sigma_1 C} \left[\varphi^{(1)}(x, y) - \sigma_1 \mathcal{P}_{j_1}^{(1)}(\bar{x} - x, y) \right], \\ \psi^{(2)}(z) &:= \frac{1}{1 + \sigma_2 C} \left[\varphi^{(2)}(z) - \sigma_2 \mathcal{P}_{k_1}^{(2)}(z - \bar{y}) \right], \end{aligned}$$

where $j_1 := \lfloor \log_2 t \rfloor + 1$ and $k_1 := \lfloor \log_2 s \rfloor + 1$.

Then for the appropriate constant C , the function $\psi(x, y) = \int_{\mathbb{R}^m} \psi^{(1)}(x, y - z) \psi^{(2)}(z) dz$ is in $\mathcal{M}_{flag}(1, 1, t, s, \bar{x}, \bar{y})$ with $\|\psi\|_{\mathcal{M}_{flag}(1, 1, t, s, \bar{x}, \bar{y})} \leq 1$.

Based on the definition of ψ , we have

$$\begin{aligned} |\langle f, \varphi \rangle| &= \left| \int_{\mathbb{R}^{n+m}} f(x, y) \varphi(x, y) dx dy \right| \\ &= \left| \int_{\mathbb{R}^{n+m}} f(x, y) \int_{\mathbb{R}^m} \varphi^{(1)}(x, y - z) \varphi^{(2)}(z) dz dx dy \right| \\ &= \left| \int_{\mathbb{R}^{n+m}} f(x, y) \int_{\mathbb{R}^m} \left[(1 + \sigma_1 C) \psi^{(1)}(x, y - z) + \sigma_1 \mathcal{P}_{j_1}^{(1)}(\bar{x} - x, y - z) \right] \right. \\ &\quad \times \left. \left[(1 + \sigma_2 C) \psi^{(2)}(z) + \sigma_2 \mathcal{P}_{k_1}^{(2)}(z - \bar{y}) \right] dz dx dy \right| \\ &\leq \left| \int_{\mathbb{R}^{n+m}} f(x, y) \int_{\mathbb{R}^m} (1 + \sigma_1 C) \psi^{(1)}(x, y - z) (1 + \sigma_2 C) \psi^{(2)}(z) dz dx dy \right| \\ &\quad + \left| \int_{\mathbb{R}^{n+m}} f(x, y) \int_{\mathbb{R}^m} \sigma_1 \mathcal{P}_{j_1}^{(1)}(\bar{x} - x, y - z) (1 + \sigma_2 C) \psi^{(2)}(z) dz dx dy \right| \\ &\quad + \left| \int_{\mathbb{R}^{n+m}} f(x, y) \int_{\mathbb{R}^m} (1 + \sigma_1 C) \psi^{(1)}(x, y - z) \sigma_2 \mathcal{P}_{k_1}^{(2)}(z - \bar{y}) dz dx dy \right| \\ &\quad + \left| \int_{\mathbb{R}^{n+m}} f(x, y) \int_{\mathbb{R}^m} \sigma_1 \mathcal{P}_{j_1}^{(1)}(\bar{x} - x, y - z) \sigma_2 \mathcal{P}_{k_1}^{(2)}(z - \bar{y}) dz dx dy \right| \\ &=: A_1 + A_2 + A_3 + A_4. \end{aligned}$$

For the term A_1 , from (3.8) we obtain that

$$A_1 \leq C \left(M_1 \left(M_2(|u^+|^r) \right) (\bar{x}, \bar{y}) \right)^{\frac{1}{r}}$$

For the term A_4 , by definition we have

$$A_4 \leq C u^+(\bar{x}, \bar{y}) = C \left(|u^+(\bar{x}, \bar{y})|^r \right)^{\frac{1}{r}} \leq C \left(M_1 \left(M_2(|u^+|^r) \right) (\bar{x}, \bar{y}) \right)^{\frac{1}{r}}.$$

As for A_2 , we write

$$\begin{aligned} A_2 &= \left| \int_{\mathbb{R}^m} \int_{\mathbb{R}^{n+m}} f(x, y) \sigma_1 P_{j_1}^{(1)}(\bar{x} - x, y - z) dx dy (1 + \sigma_2 C) \psi^{(2)}(z) dz \right| \\ &= \left| \int_{\mathbb{R}^m} F_{\bar{x}, j_1}(z) (1 + \sigma_2 C) \psi^{(2)}(z) dz \right|, \end{aligned}$$

where

$$F_{\bar{x}, j_1}(z) := \int_{\mathbb{R}^{n+m}} f(x, y) \sigma_1 P_{j_1}^{(1)}(\bar{x} - x, y - z) dx dy.$$

Then following the same approach as above, by using the reproducing formula in terms of $Q_k^{(2)}$, the almost orthogonality estimates, we obtain that

$$\begin{aligned} A_2 &\leq C \left(M_2 \left(\sup_{s>0} \left| \int_{\mathbb{R}^m} F_{\bar{x}, j_1}(z) P_s^{(2)}(z) dz \right|^r \right) (\bar{y}) \right)^{\frac{1}{r}} \\ &\leq C \left(M_2 \left(\sup_{s>0} \left| \int_{\mathbb{R}^m} \int_{\mathbb{R}^{n+m}} f(x, y) \sigma_1 P_{j_1}^{(1)}(\bar{x} - x, y - z) dx dy P_s^{(2)}(z) dz \right|^r \right) (\bar{y}) \right)^{\frac{1}{r}} \\ &\leq C \left(M_2 \left(M_1(|u^+|^r) \right) (\bar{x}, \bar{y}) \right)^{\frac{1}{r}}. \end{aligned}$$

Symmetrically, we obtain that

$$A_3 \leq C \left(M_1 \left(M_2(|u^+|^r) \right) (\bar{x}, \bar{y}) \right)^{\frac{1}{r}}.$$

Combining the estimates of A_1 , A_2 , A_3 and A_4 , we obtain that (3.7) holds.

3.3 The estimate $\|u^+\|_1 \lesssim \sum_{j=1}^{n+m} \sum_{k=1}^m \|R_{j,k}(f)\|_1 + \|f\|_1$

Let $P_t^{(1)}$ be the Poisson kernel on \mathbb{R}^{n+m} and $Q_{j,t}^{(1)}$ to be the j -th conjugate Poisson kernel on \mathbb{R}^{n+m} . Then following [4, Section 8], it is easy to verify that $u := u_0^{(1)} = P_t^{(1)} * f$, $u_j^{(1)} = Q_{j,t}^{(1)} * f = P_t^{(1)} * (R_j^{(1)} * f)$, $j = 1, 2, \dots, n+m$ is a $(n+m+1)$ -tuple of harmonic functions that satisfy the equations of conjugacy:

$$\begin{cases} \frac{\partial u_j^{(1)}}{\partial x_j} = \frac{\partial u_i^{(1)}}{\partial x_j}, & 0 \leq i, j \leq n+m; \\ \sum_{j=0}^{n+m} \frac{\partial u_j^{(1)}}{\partial x_j} = 0. \end{cases} \quad (3.11)$$

Here we use $R_j^{(1)}$ to denote the j th Riesz transform on \mathbb{R}^{n+m} , $j = 1, 2, \dots, n+m$. Similarly, we use $P_s^{(2)}$ to denote the Poisson kernel on \mathbb{R}^m and $Q_{k,s}^{(2)}$ to denote the k -th conjugate Poisson kernel on \mathbb{R}^m .

Again, following [4, Section 8], we can verify that $u := u_0^{(2)} = P_s^{(2)} *_{\mathbb{R}^m} f$, $u_k^{(2)} = Q_{k,s}^{(2)} *_{\mathbb{R}^m} f = P_s^{(2)} *_{\mathbb{R}^m} (R_k^{(2)} *_{\mathbb{R}^m} f)$, $k = 1, 2, \dots, m$ is a $(m+1)$ -tuple of harmonic functions that satisfy the equations of conjugacy:

$$\begin{cases} \frac{\partial u_j^{(2)}}{\partial x_j} = \frac{\partial u_i^{(2)}}{\partial x_j}, & 0 \leq i, j \leq m; \\ \sum_{j=0}^m \frac{\partial u_j^{(2)}}{\partial x_j} = 0. \end{cases} \quad (3.12)$$

Here we use $R_k^{(2)}$ to denote the k th Riesz transform on \mathbb{R}^m , $k = 1, 2, \dots, m$.

We now set $u(x, y, t, s) = u_{0,0}(x, y, t, s) = P_t^{(1)} *_{\mathbb{R}^m} P_s^{(2)} * f(x, y)$. Then we define

$$u_{1,0}(x, y, t, s) = Q_{1,t}^{(1)} *_{\mathbb{R}^m} P_s^{(2)} * f(x, y) \quad \text{and} \quad u_{0,1}(x, y, t, s) = P_t^{(1)} *_{\mathbb{R}^m} Q_{1,s}^{(2)} * f(x, y),$$

and similarly,

$$u_{j,k}(x, y, t, s) = Q_{j,t}^{(1)} *_{\mathbb{R}^m} Q_{k,s}^{(2)} * f(x, y),$$

for $j = 1, \dots, n+m$ and $k = 1, \dots, m$.

We first point out that for $k = 1, \dots, m$, the tuple $(u_{0,k}, u_{1,k}, \dots, u_{n+m,k})$ satisfies the Cauchy–Riemann equation in (3.11), and that for $j = 1, \dots, n+m$ the tuple $(u_{j,0}, u_{j,1}, \dots, u_{j,m})$ satisfies the Cauchy–Riemann equation in (3.12).

Following the idea in [4, Section 8], we consider the matrix valued function

$$F = \begin{bmatrix} u_{0,0} & \dots & u_{0,m} \\ \dots & \dots & \dots \\ u_{n+m,0} & \dots & u_{n+m,m} \end{bmatrix} = P_t^{(1)} *_{\mathbb{R}^m} P_s^{(2)} * \tilde{F},$$

where we denote

$$\tilde{F} = \begin{bmatrix} f & \dots & R_m^{(2)} *_{\mathbb{R}^m} f \\ \dots & \dots & \dots \\ R_{m+n}^{(1)} * f & \dots & R_{m+n}^{(1)} *_{\mathbb{R}^m} R_m^{(2)} * f \end{bmatrix}.$$

We obtain

$$\begin{aligned}
& \sup_{t>0} \sup_{s>0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |F(x, y, t, s)| dx dy \\
& \leq \sup_{t>0} \sup_{s>0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \left(\sum_{j=0}^{n+m} \sum_{k=0}^m |u_{j,k}(x, y, t, s)|^2 \right)^{\frac{1}{2}} dx dy \\
& \leq C \sum_{j=0}^{n+m} \sum_{k=0}^m \sup_{t>0} \sup_{s>0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \left| Q_{j,t}^{(1)} *_{\mathbb{R}^m} Q_{k,s}^{(2)} * f(x, y) \right| dx dy \\
& \leq C \sum_{j=0}^{n+m} \sum_{k=0}^m \sup_{t>0} \sup_{s>0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \left| P_t^{(1)} *_{\mathbb{R}^m} P_s^{(2)} * (R_j^{(1)} * R_k^{(2)} *_{\mathbb{R}^m} f)(x, y) \right| dx dy \\
& \leq C \sum_{j=0}^{n+m} \sum_{k=0}^m \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \left| (R_j^{(1)} *_{\mathbb{R}^m} R_k^{(2)} *)(f)(x, y) \right| dx dy,
\end{aligned}$$

where the last inequality follows from the fact that

$$\int_{\mathbb{R}^{n+m}} P_t^{(1)}(x - x_1, y - y_1) dx dy = C_{n+m} \text{ and } \int_{\mathbb{R}^m} P_s^{(2)}(y - y_1) dy = C_m$$

for all $t, s > 0$, $x_1 \in \mathbb{R}^n$ and $y_1 \in \mathbb{R}^m$.

Next it suffices to show

$$\|u^+\|_{L^1(\mathbb{R}^n \times \mathbb{R}^m)} \leq C \sup_{t>0} \sup_{s>0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |F(x, y, t, s)| dx dy. \quad (3.13)$$

To see this, we have that for $q < 1$,

$$\begin{aligned}
|F(x, y, t + \epsilon_1, s + \epsilon_2)|^q &= \left| P_{t+\epsilon_1}^{(1)} * P_{s+\epsilon_2}^{(2)} *_{\mathbb{R}^m} \tilde{F}(x, y) \right|^q = \left| P_t^{(1)} * P_{\epsilon_1}^{(1)} * P_{s+\epsilon_2}^{(2)} *_{\mathbb{R}^m} \tilde{F}(x, y) \right|^q \\
&= \left| P_t^{(1)} * F(x, y, \epsilon_1, s + \epsilon_2) \right|^q \\
&\leq C_{q,m} \sum_{k=0}^m \left| P_t^{(1)} * F_k(x, y, \epsilon_1, s + \epsilon_2) \right|^q,
\end{aligned}$$

where for each k , F_k is the k th column in the matrix F . Since $P_t^{(1)} * F_k$ satisfies the generalised Cauchy–Riemann equations in (3.11) for the variable (x, y, t) , we get that $|P_t^{(1)} * F_k|^q$ is subharmonic for $q \geq \frac{n+m-1}{n+m}$. Then from the subharmonic inequality [27, Equation (59), Section 4.2, Chapter 3] we have that for $q \geq \frac{n+m-1}{n+m}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $t > 0$ and $\epsilon_1 > 0$,

$$\left| P_t^{(1)} * F_k(x, y, \epsilon_1, s + \epsilon_2) \right|^q \leq P_t^{(1)} * |F_k(x, y, \epsilon_1, s + \epsilon_2)|^q,$$

which implies that

$$\begin{aligned}
|F(x, y, t + \epsilon_1, s + \epsilon_2)|^q &\leq C_{q,m} \sum_{k=0}^m P_t^{(1)} * |F_k(x, y, \epsilon_1, s + \epsilon_2)|^q \\
&\leq C_{q,m} P_t^{(1)} * |F(x, y, \epsilon_1, s + \epsilon_2)|^q.
\end{aligned} \quad (3.14)$$

And we use the basic fact that $|F|^q = (\sum_{k=0}^m |F_k|^2)^{\frac{q}{2}} \approx \sum_{k=0}^m |F_k|^q$.

Again, for $F(x, y, \epsilon_1, s + \epsilon_2)$, we have

$$\begin{aligned} |F(x, y, \epsilon_1, s + \epsilon_2)|^q &= |P_s^{(2)} *_{\mathbb{R}^m} F(x, y, \epsilon_1, \epsilon_2)|^q \\ &\leq C_{q,n+m} \sum_{j=0}^{n+m} \left| P_s^{(2)} *_{\mathbb{R}^m} \tilde{F}_j(x, y, \epsilon_1, \epsilon_2) \right|^q, \end{aligned}$$

where for each j , \tilde{F}_j is the j th row in the matrix F . Since $P_s^{(2)} *_{\mathbb{R}^m} \tilde{F}_j$ satisfies the generalised Cauchy–Riemann equations in (3.11) for the variable (y, s) , we get that $|P_s^{(2)} *_{\mathbb{R}^m} \tilde{F}_j|^q$ is subharmonic for $q \geq \frac{m-1}{m}$. Then again, from the subharmonic inequality [27, Equation (59), Section 4.2, Chapter 3] we have that for $q \geq \frac{m-1}{m}$, $y \in \mathbb{R}^m$, $s > 0$ and $\epsilon_2 > 0$,

$$\begin{aligned} |F(x, y, \epsilon_1, s + \epsilon_2)|^q &\leq C_{q,n+m} \sum_{j=0}^{n+m} P_s^{(2)} *_{\mathbb{R}^m} |\tilde{F}_j(x, y, \epsilon_1, \epsilon_2)|^q \\ &\leq C_{q,n+m} P_s^{(2)} *_{\mathbb{R}^m} |F(x, y, \epsilon_1, \epsilon_2)|^q. \end{aligned} \tag{3.15}$$

And we use the basic fact that $|F|^q = (\sum_{j=0}^{n+m} |\tilde{F}_j|^2)^{\frac{q}{2}} \approx \sum_{j=0}^{n+m} |\tilde{F}_j|^q$.

Combining the estimates of (3.14) and (3.15), we obtain that

$$|F(x, y, t + \epsilon_1, s + \epsilon_2)|^q \leq C_{q,n,m} P_t^{(1)} * P_s^{(2)} *_{\mathbb{R}^m} |F(x, y, \epsilon_1, \epsilon_2)|^q.$$

Then, following the convergence argument in [4, Section 8], also in [27, Section 4.2], we obtain that

$$\|u^+\|_1 \leq C_{m,n} \sup_{t>0} \sup_{s>0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |F(x, y, t, s)| dx dy.$$

which implies that the claim (3.13) holds.

4 Characterizations of the flag Hardy spaces

In this section, the following estimates will be concluded:

- (I) $\|S_F(f)\|_1 \lesssim \|S_F(u)\|_1$,
- (II) $\|u^*\|_1 \approx \|M_\Phi^*(f)\|_1$,
- (III) $\|u^+\|_1 \approx \|M_\Phi^+(f)\|_1$,
- (IV) $\sum_{j=1}^{n+m} \sum_{k=1}^m \|R_{j,k}(f)\|_1 + \|f\|_1 \lesssim \|g_F(f)\|_1$.

4.1 The estimate $\|S_F(f)\|_1 \lesssim \|S_F(u)\|_1$

The estimate $\|S_F(f)\|_1 \lesssim \|S_F(u)\|_1$, follows from the same ideas in Section 2.2 and 2.3. More precisely, we first need to establish the following discrete Calderón reproducing formula. For this purpose, let $\phi^{(1)}(x, y) \in \mathcal{S}(\mathbb{R}^{n+m})$, radial and satisfy the following conditions:

- (i) $\text{supp } \phi^{(1)} \subset B(0, 1)$, where $B(0, 1)$ is the unit ball in \mathbb{R}^{n+m} ;
- (ii) $\int_{\mathbb{R}^{n+m}} x^\alpha y^\beta \phi^{(1)}(x, y) dx dy = 0$, where $|\alpha| + |\beta| \leq 2(n \vee m)$;
- (iii) $\int_0^\infty e^{-u} \widehat{\phi^{(1)}}(u) du = -1$.

In fact, $\phi^{(1)}(x, y)$ can be constructed as follows. Let $h^{(1)} \in \mathcal{S}(\mathbb{R}^{n+m})$, radial and support in $B(0, 1)$. Let $k = 4(n \vee m)$ and $\phi^{(1)}(x, y) = \Delta^k h^{(1)}(x, y)$. By multiplying an appropriate constant, we can see that such $\phi^{(1)}(x, y)$ satisfies all the conditions above.

Similarly, Let $h^{(2)} \in \mathcal{S}(\mathbb{R}^m)$, radial and support in $B(0, 1)$ and $\phi^{(2)}(z) = \Delta^k h^{(2)}(z)$. By multiplying a proper constant, we can obtain that $\phi^{(2)}(z) \in \mathcal{S}(\mathbb{R}^m)$, radial and satisfies the following conditions:

- (i) $\text{supp } \phi^{(2)} \subset B(0, 1)$, where $B(0, 1)$ is the unit ball in \mathbb{R}^m ;
- (ii) $\int_{\mathbb{R}^m} z^\gamma \phi^{(2)}(z) dz = 0$, where $|\gamma| \leq 2(n \vee m)$;
- (iii) $\int_0^\infty e^{-u} \widehat{\phi^{(2)}}(u) du = -1$.

Let $\phi(x, y) = \phi^{(1)} *_{\mathbb{R}^m} \phi^{(2)}(x, y)$ and $\phi_{t,s}(x, y) = \phi_t^{(1)} *_{\mathbb{R}^m} \phi_s^{(2)}(x, y)$. Repeating the same proof as in Theorem 2.5, obtain the following statement.

Theorem 4.1. *There exist $\phi_{j,k,I,J}(x, y) \in \mathcal{M}_{flag}(\beta, \gamma, 2^{-j}, 2^{-k}, x_I, y_J)$ and a fixed large integer N such that*

$$\begin{aligned} & f(x, y) \\ &= \sum_j \sum_k \sum_I \sum_J |I||J| \phi_{j,k,I,J}(x, y) \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \phi_{t,s} * \left(ts \frac{\partial}{\partial t} \frac{\partial}{\partial s} P_{t,s} \right) * f(x_I, y_J) \frac{dt}{t} \frac{ds}{s}, \end{aligned}$$

where $I \subset \mathbb{R}^n$ and $J \subset \mathbb{R}^m$ are dyadic cubes with side-lengths $\ell(I) = 2^{-j-N}$ and $\ell(J) = 2^{-((j-N) \wedge (k-N))}$, x_I and y_J are **any fixed points** in I and J , respectively. Moreover, for $f \in L^1(\mathbb{R}^{n+m})$ and ψ is the same as in (1.1),

$$\begin{aligned} & \langle f, \psi \rangle \\ &= \left\langle \sum_j \sum_k \sum_I \sum_J |I||J| \phi_{j,k,I,J}(\cdot, \cdot) \int_{2^{-k-N}}^{2^{-k-N+1}} \int_{2^{-j-N}}^{2^{-j-N+1}} \left(ts \frac{\partial}{\partial t} \frac{\partial}{\partial s} P_{t,s} \right) * f(x_I, y_J) \frac{dt}{t} \frac{ds}{s}, \psi \right\rangle. \end{aligned}$$

Applying the same proof as in Section 2.2 gives the following.

Theorem 4.2. *Let $f \in L^1(\mathbb{R}^{n+m})$, we have*

$$\begin{aligned} & \left\| \sum_j \sum_k \sum_I \sum_J \int_{2^{-j}}^{2^{-j+1}} \int_{2^{-k}}^{2^{-k+1}} \sup_{u \in I, v \in J} |\psi_{t,s} * f(u, v)|^2 \frac{dt}{t} \frac{ds}{s} \chi_I(x) \chi_J(y) \right\|_1 \\ & \approx \left\| \sum_j \sum_k \sum_I \sum_J \int_{2^{-j}}^{2^{-j+1}} \int_{2^{-k}}^{2^{-k+1}} \inf_{u \in I, v \in J} \left| \left(ts \frac{\partial}{\partial t} \frac{\partial}{\partial s} P_{t,s} \right) * f(u, v) \right|^2 \frac{dt}{t} \frac{ds}{s} \chi_I(x) \chi_J(y) \right\|_1, \end{aligned}$$

where $I \subset \mathbb{R}^n$ and $J \subset \mathbb{R}^m$ are dyadic cubes with side-lengths $\ell(I) = 2^{-j}$ and $\ell(J) = 2^{-(j \wedge k)}$ and the lower left-corners $l_1 2^{-j}$ and $l_2 2^{-(j \wedge k)}$, respectively.

The estimate $\|S_F(f)\|_1 \lesssim \|S_F(u)\|_1$, then follows from Theorem 4.2 as in Section 2.3. We leave the details to the reader.

4.2 The equivalence $\|u^*\|_1 \approx \|M_\Phi^*(f)\|_1$

We first show

$$\|u^*\|_1 \leq C \|M_\Phi^*(f)\|_1.$$

To do this, we introduce the “tangential” maximal function M_N^{**} (depending on a parameter N) by

$$M_N^{**}(f)(x, y) = \sup_{u \in \mathbb{R}^n, v \in \mathbb{R}^m, t, s > 0} |f * \phi_{t,s}(x - u, y - v)| \frac{1}{\left(1 + \frac{|u|}{t}\right)^N \left(1 + \frac{|v|}{t+s}\right)^N}.$$

Obviously,

$$M_\phi^+(f)(x, y) \leq M_\phi^*(f)(x, y) \leq 2^{2N} M_N^{**}(f)(x, y).$$

Next, we introduce the grand maximal functions. For this purpose, we first note that on $\mathcal{S}(\mathbb{R}^{n+m})$ one has a denumerable collection of seminorms $\|\cdot\|_{\alpha_1, \alpha_2, \beta_1, \beta_2}$ given by

$$\|\phi\|_{\alpha_1, \alpha_2, \beta_1, \beta_2} = \sup_{(x, y) \in \mathbb{R}^{n+m}} \left| x^{\alpha_1} y^{\alpha_2} \partial_x^{\beta_1} \partial_y^{\beta_2} \phi(x, y) \right|.$$

Similarly, on $\mathcal{S}(\mathbb{R}^m)$, seminorms $\|\cdot\|_{\alpha, \beta}$ are given by

$$\|\phi\|_{\alpha, \beta} = \sup_{z \in \mathbb{R}^m} \left| z^\alpha \partial_z^\beta \phi(z) \right|.$$

Let $\mathcal{F}^{(1)} = \{\|\cdot\|_{\alpha_1^i, \alpha_2^i, \beta_1^i, \beta_2^i}\}$ be any finite collections of seminorms on $\mathcal{S}(\mathbb{R}^{n+m})$ and $\mathcal{F}^{(2)} = \{\|\cdot\|_{\alpha^i, \beta^i}\}$ be any finite collections of seminorms on $\mathcal{S}(\mathbb{R}^m)$. Set

$$\mathcal{F} = \left\{ \phi \in \mathcal{D}_F(\mathbb{R}^n \times \mathbb{R}^m) : \text{for all } \phi^\sharp \in \mathcal{S}(\mathbb{R}^{n+m} \times \mathbb{R}^m) \text{ satisfying } \phi(x, y) = \int_{\mathbb{R}^m} \phi^\sharp(x, y - z, z) dz, \right.$$

$$\|\phi^\sharp(\cdot, \cdot, z)\|_{\alpha_1, \alpha_2, \beta_1, \beta_2} \leq 1 \text{ for all } z \in \mathbb{R}^m \text{ and } \|\cdot\|_{\alpha_1, \alpha_2, \beta_1, \beta_2} \in \mathcal{F}^{(1)}; \\ \|\phi^\sharp(x, y, \cdot)\|_{\alpha, \beta} \leq 1 \text{ for all } (x, y) \in \mathbb{R}^{n+m} \text{ and } \|\cdot\|_{\alpha, \beta} \in \mathcal{F}^{(2)}.\}$$

We then define

$$M_{\mathcal{F}}(f)(x, y) = \sup_{\phi \in \mathcal{F}} M_{\phi}^+(f)(x, y).$$

We need the following results.

Lemma 4.3. *If $M_{\phi}^*(f) \in L^1(\mathbb{R}^{n+m})$ and $N > 2(n \vee m)$, then $M_N^{**}(f) \in L^1(\mathbb{R}^{n+m})$ with*

$$\|M_N^{**}(f)\|_1 \leq C_{N,p} \|M_{\phi}^*(f)\|_1. \quad (4.1)$$

We point out that if

$$M_{\phi, a, b}^*(f)(x, y) = \sup_{(x_1, y_1, t, s) \in \Gamma_{a, b}(x, y)} |\phi_{t, s} * f(x_1, y_1)|,$$

where $\Gamma_{a, b}(x, y) = \{(x_1, y_1, t, s) : |x - x_1| \leq at, |y - y_1| \leq b(t + s)\}$, then

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} |M_{\phi, a, b}^*(f)(x, y)|^p dx dy \leq C_{n, m} (1 + a)^n (1 + b)^m \int_{\mathbb{R}^n \times \mathbb{R}^m} |M_{\phi}^*(f)(x, y)|^p dx dy. \quad (4.2)$$

This can be obtained by mimicking the proof in [27, §2.5, Chapter 2]. Observing

$$\frac{|f * \phi_{t, s}(x - u, y - v)|}{\left(1 + \frac{|u|}{t}\right)^N \left(1 + \frac{|v|}{t + s}\right)^N} \leq \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} 2^{(1-k)N} 2^{(1-\ell)N} |M_{\phi, 2^{k+1}, 2^{\ell+1}}^*(f)(x, y)|$$

for all $u \in \mathbb{R}^n, v \in \mathbb{R}^m, t, s > 0$ and $N > 0$, and using (4.2), we then get (4.1) with

$$C_N^p = c_{n, m} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} (1 + 2^k)^n \cdot (1 + 2^{\ell})^m \cdot 2^{(1-k)N} \cdot 2^{(1-\ell)N},$$

which is finite if $N > 2(n \vee m)$. The proof of the Lemma 4.3 is concluded.

Next we recall the following lemma from [27] which will be used to pass from one approximation of the identity to another.

Lemma 4.4. [27, Lemma 2, §1.3] *Suppose we are given ϕ and $\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \phi = 1$. Then there is a sequence $\{\eta^{(k)}\} \subset \mathcal{S}(\mathbb{R}^d)$ so that*

$$\psi = \sum_{k=0}^{\infty} \eta^{(k)} * \phi_{2^{-k}} \quad (4.3)$$

with $\eta^{(k)} \rightarrow 0$ rapidly, in the sense that whenever $\|\cdot\|_{\alpha, \beta}$ is a seminorm and $M \geq 0$ is fixed, then

$$\|\eta^{(k)}\|_{\alpha, \beta} = O(2^{-kM}) \quad \text{as } k \rightarrow \infty.$$

From Lemma 4.4, we obtain the following estimate

$$\|M_{\mathcal{F}}(f)\|_1 \leq C\|M_{\phi}^*(f)\|_1. \quad (4.4)$$

Indeed, for any $\phi = \phi^{(1)} *_{\mathbb{R}^m} \phi^{(2)} \in \mathcal{S}_F(\mathbb{R}^n \times \mathbb{R}^m)$, by (4.3) on $\phi^{(1)}$ and $\phi^{(2)}$ we have

$$\begin{aligned} M_{\phi}(f)(x, y) &\leq \sup_{t, s > 0} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \left| f * (\phi_{2^{-k}t}^{(1)} *_{\mathbb{R}^m} \phi_{2^{-\ell}s}^{(2)}) * (\eta_t^{(1),(k)} *_{\mathbb{R}^m} \eta_s^{(2),(\ell)}) \right| \\ &\leq M_N^{**}(f)(x, y) \sup_{t, s > 0} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \int_{\mathbb{R}^n \times \mathbb{R}^m} \left(1 + \frac{|u|}{2^{-k}t}\right)^N \left(1 + \frac{|v|}{2^{-\ell}(t+s)}\right)^N \\ &\quad \times |\eta_t^{(1),(k)} *_{\mathbb{R}^m} \eta_s^{(2),(\ell)}(u, v)| dudv \\ &\leq CM_N^{**}(f)(x, y), \end{aligned}$$

where the last inequality holds if ϕ belongs to an appropriate chosen \mathcal{F} . Thus

$$M_{\mathcal{F}}(f)(x, y) = \sup_{\phi \in \mathcal{F}} M_{\phi}^+(f)(x, y) \leq CM_N^{**}(f)(x, y)$$

for all $x \in \mathbb{R}^n, y \in \mathbb{R}^m$; taking $N > 2(n \vee m)$ as in (4.1) yields (4.4).

Next, we will show that

$$\|M_{\phi}^*(f)\|_1 \leq C\|M_{\phi}^+(f)\|_1. \quad (4.5)$$

Let \mathcal{F} be the same as in (4.4) and for any fixed $\lambda > 0$, let

$$F = F_{\lambda} = \{(x, y) : M_{\mathcal{F}}(f)(x, y) \leq \lambda M_{\phi}^*(f)(x, y)\}.$$

We prove (4.5) by showing that, for any $q > 0$,

$$M_{\phi}^*(f)(x, y) \leq C[M_s(M_{\phi}^+(f))^q]^{\frac{1}{q}} \quad \text{for } (x, y) \in F,$$

where M_s is the strong maximal function. Now for any (x, y) , there exists (x_1, y_1, t, s) with $|x - x_1| < t$, $|y - y_1| < t + s$ and $f * \phi_{t,s}(x_1, y_1) \geq \frac{1}{2}M_{\phi}^*(f)(x, y)$. Choose r_1 small and consider the ball centered at x_1 of radius $r_1 t$, i.e. the points u so that $|x_1 - u| < r_1 t$. We have that

$$|f * \phi_{t,s}(x_1, y_1) - f * \phi_{t,s}(u, y_1)| \leq r_1 t \sup_{|u - x_1| < r_1 t} |\nabla_u f * \phi_{t,s}(u, y_1)|.$$

Similarly, choose r_2 small and consider the ball centered at y_1 of radius $r_2(t + s)$, i.e. the points v so that $|y_1 - v| < r_2(t + s)$. We have that

$$|f * \phi_{t,s}(x_1, y_1) - f * \phi_{t,s}(x_1, v)| \leq r_2(t + s) \sup_{|v - y_1| < r_2(t + s)} |\nabla_v f * \phi_{t,s}(x_1, v)|.$$

Combining the above two case, we have

$$|f * \phi_{t,s}(x_1, y_1) - f * \phi_{t,s}(u, y_1) - f * \phi_{t,s}(x_1, v) + f * \phi_{t,s}(u, v)|$$

$$\leq Cr_1 t \cdot r_2(t+s) \sup_{|u-x_1| < r_1 t, |v-y_1| < r_2(t+s)} |\nabla_u \nabla_v f * \phi_{t,s}(u, v)|.$$

However, $\frac{\partial}{\partial u_i} f * \phi_{t,s}(u, v) = f * \tilde{\phi}_{t,s}^i(u, v)$, where

$$\tilde{\phi}_{t,s}^i(u, v) = \int_{\mathbb{R}^m} \frac{\partial}{\partial u_i} \phi_t^{(1)}(u, v-w) \phi_s^{(2)}(w) dw = \frac{1}{t} \int_{\mathbb{R}^m} \left(\frac{\partial}{\partial u_i} \phi^{(1)} \right)_t(u, v-w) \phi_s^{(2)}(w) dw.$$

And $\frac{\partial}{\partial v_j} f * \phi_{t,s}(u, v) = f * \bar{\phi}_{t,s}^j(u, v)$, where

$$\begin{aligned} \bar{\phi}_{t,s}^j(u, v) &= \frac{1}{t} \int_{\mathbb{R}^m} \left(\frac{\partial}{\partial v_j} \phi^{(1)} \right)_t(u, v-w) \phi_s^{(2)}(w) dw \quad \text{if } t > s; \\ \bar{\phi}_{t,s}^j(u, v) &= \frac{1}{s} \int_{\mathbb{R}^m} \phi_t^{(1)}(u, w) \left(\frac{\partial}{\partial v_j} \phi^{(2)} \right)_s(v-w) dw \quad \text{if } t \leq s. \end{aligned}$$

Note that the set of functions of the form $\tilde{\phi}^i(x+h_1, y+h_2)$ and $\bar{\phi}^j(x+h_1, y+h_2)$, $|h_1| \leq 1+r_1$, $|h_2| \leq 1+r_2$, $i = 1, \dots, n$, $j = 1, \dots, m$, is a compact set in $\mathcal{S}_F(\mathbb{R}^n \times \mathbb{R}^m)$, hence we have $c\tilde{\phi}^i(x+h_1, y+h_2)$ and $c\bar{\phi}^j(x+h_1, y+h_2) \in \mathcal{F}$, where c is a constant independent of ϕ , h_1 and h_2 . Thus $|f * \phi_{t,s}(x_1, y_1) - f * \phi_{t,s}(u, y_1)| \leq cr_1 M_{\mathcal{F}}(f)(x, y) \leq cr_1 \lambda M_{\phi}^*(f)(x, y)$, if $(x, y) \in F$.

By considering the case $t > s$, if $(x, y) \in F$ then we can obtain that

$$|f * \phi_{t,s}(x_1, y_1) - f * \phi_{t,s}(x_1, v)| \leq cr_2 \lambda M_{\phi}^*(f)(x, y),$$

and that

$$|f * \phi_{t,s}(x_1, y_1) - f * \phi_{t,s}(u, y_1) - f * \phi_{t,s}(x_1, v) + f * \phi_{t,s}(u, v)| \leq Cr_1 \cdot r_2 \lambda M_{\phi}^*(f)(x, y)$$

So if we take r_1 and r_2 so small that $cr_1 \lambda$, $cr_2 \lambda$, $cr_1 r_2 \lambda < 1/16$, then we have

$$|f * \phi_{t,s}(u, v)| > \frac{1}{4} M_{\phi}^*(f)(x, y) \quad \text{for all } u \in B(x_1, r_1 t) \text{ and } v \in B(y_1, r_2 t).$$

Thus we get that

$$\begin{aligned} \frac{1}{4^q} |M_{\phi}^*(f)(x, y)|^q &\leq \frac{1}{|B(x_1, r_1 t)| \times |B(y_1, r_2 t)|} \int_{B(x_1, (1+r_1)t) \times B(y_1, (1+r_2)t)} |f * \phi_{t,s}(u, v)|^q du dv \\ &\leq \left(\frac{1+r_1}{r_1} \right)^n \left(\frac{1+r_2}{r_2} \right)^m M_s[(M_{\phi}^+(f))^q](x, y), \end{aligned}$$

which is (4.5). Similarly, we can obtain this result when considering the case $t \leq s$.

Then using the maximal theorem (for M_s) with $q < 1$ leads to

$$\int_F M_{\phi}^*(f)(x, y) dx dy \leq C \int_{\mathbb{R}^n \times \mathbb{R}^m} (M_s[(M_{\phi}^+(f))^q](x, y))^{\frac{1}{q}} dx dy \leq C \int_{\mathbb{R}^n \times \mathbb{R}^m} M_{\phi}(f)(x, y) dx dy. \quad (4.6)$$

Now we claim that

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} M_{\phi}^*(f)(x, y) dx dy \leq 2 \int_F M_{\phi}^*(f)(x, y) dx dy. \quad (4.7)$$

To see this, observe that

$$\int_{F^c} M_\phi^*(f)(x, y) dx dy \leq \lambda^{-1} \int_{F^c} M_{\mathcal{F}}(f)(x, y) dx dy \leq c\lambda^{-1} \int_{\mathbb{R}^n \times \mathbb{R}^m} M_\phi^*(f)(x, y) dx dy,$$

where the last inequality follows from (4.4). Thus, if we take $\lambda \geq 2c$, we verify the claim (4.7), which, together with (4.6), yields (4.5).

We recall the result that if $P^{(1)}(x, y)$ is the Poisson kernel on \mathbb{R}^{n+m} , then

$$P^{(1)}(x, y) = \frac{c_{n+m}}{(1 + |x|^2 + |y|^2)^{(n+m+1)/2}} = \sum_{k=0}^{\infty} 2^{-k} \phi_{2^k}^{(1),(k)}(x, y),$$

where $\{\phi^{(1),(k)}\}$ is a bounded collection of functions in $\mathcal{S}(\mathbb{R}^{n+m})$. Similarly, if $P^{(1)}(z)$ is the Poisson kernel on \mathbb{R}^{n+m} , then

$$P^{(2)}(z) = \frac{c_m}{(1 + |z|^2)^{(m+1)/2}} = \sum_{\ell=0}^{\infty} 2^{-\ell} \phi_{2^\ell}^{(2),(\ell)}(x, y),$$

where $\{\phi^{(2),(\ell)}\}$ is a bounded collection of functions in $\mathcal{S}(\mathbb{R}^m)$. Then for the Poisson kernel $P_{t,s}(x, y)$, we have that

$$P_{t,s}(x, y) = P_t^{(1)} *_{\mathbb{R}^m} P_s^{(2)}(x, y) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} 2^{-k} 2^{-\ell} \phi_{2^k t}^{(1),(k)} *_{\mathbb{R}^m} \phi_{2^\ell s}^{(2),(\ell)}(x, y),$$

where obviously, $\{\phi_{2^k t, 2^\ell s}^{(k),(\ell)}\} = \{\phi_{2^k t}^{(1),(k)} *_{\mathbb{R}^m} \phi_{2^\ell s}^{(2),(\ell)}\}$ is a bounded collection of functions in $\mathcal{D}_F(\mathbb{R}^n \times \mathbb{R}^m)$. Thus, we have

$$\begin{aligned} \|u^*\|_1 &\leq \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} 2^{-k} 2^{-\ell} \|M_{\phi_{2^k t, 2^\ell s}^{(k),(\ell)}}^*\|_1 \\ &\leq C \|M_{\mathcal{F}}(f)\|_1 \\ &\leq C \|M_\Phi^*(f)\|_1. \end{aligned}$$

We now prove

$$\|M_\Phi^*(f)\|_1 \leq C \|u^*\|_1.$$

Following [27, Chapter III, § 1.7], for the Poisson kernel $P_t^{(1)}(x, y)$, there exists a functions $\eta^{(1)}$ defined on $(1, \infty)$ such that

$$\int_1^\infty \eta^{(1)}(s) ds = 1, \quad \text{and} \quad \int_1^\infty s^k \eta^{(1)}(s) ds = 0, \quad k = 1, 2, \dots$$

We now set

$$\Phi^{(1)}(x, y) := \int_1^\infty \eta^{(1)}(t) P_t^{(1)}(x, y) dt.$$

Similarly, for the Poisson kernel $P_t^{(2)}(z)$, there exists a functions $\eta^{(2)}$ defined on $(1, \infty)$ such that

$$\int_1^\infty \eta^{(2)}(s)ds = 1, \quad \text{and} \quad \int_1^\infty s^k \eta^{(2)}(s)ds = 0, \quad k = 1, 2, \dots$$

We now set

$$\Phi^{(2)}(z) := \int_1^\infty \eta^{(1)}(s)P_s^{(2)}(z)ds.$$

Then we have $\Phi^{(1)}(x, y) \in \mathcal{S}(\mathbb{R}^{n+m})$ and $\Phi^{(2)}(z) \in \mathcal{S}(\mathbb{R}^m)$. Moreover, we have

$$\int_{\mathbb{R}^{n+m}} \Phi^{(1)}(x, y)dx dy = \int_1^\infty \eta^{(1)}(t)dt = 1$$

and

$$\int_{\mathbb{R}^m} \Phi^{(2)}(z)dz = \int_1^\infty \eta^{(2)}(s)ds = 1.$$

Hence, define

$$\tilde{\Phi}(x, y) = \Phi^{(1)} *_{\mathbb{R}^m} \Phi^{(2)}(x, y),$$

then we obtain that

$$M_{\tilde{\Phi}}^*(f)(x, y) \leq u^*(x, y) \int_1^\infty \eta(t)^{(1)}dt \int_1^\infty \eta^{(2)}(s)ds = u^*(x, y).$$

As a consequence, we obtain that for arbitrary $\Phi \in \mathcal{D}_F(\mathbb{R}^n \times \mathbb{R}^m)$,

$$\|M_{\Phi}^*(f)\|_1 \leq C \|M_{\tilde{\Phi}}^*(f)\|_1 \leq \|u^*\|_1.$$

4.3 The equivalence $\|u^+\|_1 \approx \|M_{\Phi}^+(f)\|_1$

It is clear that $u^+(x) \leq u^*(x)$ for $x \in \mathbb{R}^n$. By Section 4.2, $\|u^*\|_1 \lesssim \|M_{\Phi}^*(f)\|_1$ and (4.5), we have

$$\|u^+\|_1 \lesssim \|M_{\Phi}^+(f)\|_1.$$

On the other hand, by the estimates $\|u^*\|_1 \lesssim \|u^+\|_1$ and $\|M_{\Phi}^*(f)\|_1 \lesssim \|u^*\|_1$, we get

$$\|M_{\Phi}^+(f)\|_1 \lesssim \|M_{\Phi}^*(f)\|_1 \lesssim \|u^+\|_1.$$

4.4 The estimate $\sum_{j=1}^{n+m} \sum_{k=1}^m \|R_{j,k}(f)\|_1 + \|f\|_1 \lesssim \|g_F(f)\|_1$

It suffices to prove that there exists a positive constant C such that for $j = 0, 1, \dots, n+m, k = 0, \dots, m$

$$\|R_{j,k}(f)\|_1 \leq C \|f\|_{H_F^1(\mathbb{R}^n \times \mathbb{R}^m)}. \quad (4.8)$$

Indeed, as mentioned, $R_{j,k}$ is the composition of $R_j^{(1)}$ and $R_k^{(2)}$, and hence $R_{j,k}$ is bounded on $L^p(\mathbb{R}^{n+m}), 1 < p < \infty$. Therefore, the Fourier transform of $R_{j,k}$, as a distribution, is given by

$$-\frac{\xi_j}{(|\xi|^2 + |\eta|^2)^{\frac{1}{2}}} \frac{\eta_k}{|\eta|}.$$

By a result in [23], $R_{j,k}$ is a flag singular integral. Applying [17, Theorem 22 and Theorem 23] gives that $R_{j,k}$ is bounded on $H_F^1(\mathbb{R}^n \times \mathbb{R}^m)$ and then is bounded from $H_F^1(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^1(\mathbb{R}^{n+m})$, i.e., (4.8) holds.

Now we conclude, based on all estimates provided from Section 2 to Section 4, that

$$\begin{aligned}
\|S_F(f)\|_1 &\lesssim \|S_F(u)\|_1 \lesssim \|u^*\|_1 \lesssim \|M_\Phi^*(f)\|_1 \lesssim \|u^*\|_1 \\
&\lesssim \|u^+\|_1 \lesssim \|M_\Phi^+(f)\|_1 \lesssim \|u^+\|_1 \\
&\lesssim \sum_{j=1}^{n+m} \sum_{k=1}^m \|R_{j,k}(f)\|_1 + \|f\|_1 \\
&\lesssim \|g_F(f)\|_1 \\
&\lesssim \|S_F(f)\|_1.
\end{aligned}$$

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